Skew idempotent functionals of ordered semirings.

S.V. Ludkovsky.

21 December 2012

Abstract

Skew idempotent functionals of ordered semirings are studied. Different associative and non-associative semirings are considered. Theorems about properties of skew idempotent functionals are proved. Examples are given. [1]

1 Introduction.

Idempotent mathematics being a new branch has attracted much attention in recent times as a theoretical tool having important applications in mathematics and quantum physics (see [7, 12, 15] and references therein). On the other hand, functionals and measures are also used for studies of representations of groups and algebras (see, for example, [11, 2, 3, 8, 9] and references therein). Idempotent mathematics arise naturally from the consideration of the quantization and the Plank constant in physics. Earlier idempotent functionals associative and commutative relative to the operation $\odot$ and $+$ were investigated and particularly on spaces of continuous real-valued functions on compact Hausdorff spaces (see [7, 12, 15] and references therein).

Apart from previous works, in this article skew idempotent functionals generally non-associative and noncommutative relative to the operation $\odot$ are investigated (see Section 3). The axiomatic in some respect is different

[1] AMS 2010 Mathematics Subject Classification: 08A99, 16Y60, 17A01, 18A30
key words: idempotent functional, skew, semiring, quasiring

from the real case. Moreover, the compactness condition of topological spaces on which mappings and functionals are defined is dropped and the consideration is purely algebraic below. Homogeneous idempotent functionals are also considered. They may be with values in semirings or quasirings noncommutative or non-associative relative to the addition or the multiplication. In Section 2 ordered semirings are described, propositions and theorems about their construction are proved. Skew idempotent functionals on them are presented in Section 3. Their categories are studied. Semirings of functionals and their sequences are investigated as well.

Skew idempotent functionals on semirings may be used for studies of structures of semirings, their homomorphisms and representations. The main results are Propositions 2.3, 3.17, 3.19, 3.21, 3.24, 3.27, 3.28, 3.30, 3.34, 3.38, Theorems 2.5, 2.10, 2.11, 3.33, 3.39, 3.40 and Corollary 2.15.

All main results of this paper are obtained for the first time.

2 Ordered non-associative semirings

To avoid misunderstandings we first present our definitions.

1. Definitions. Let K be a non-void set. If K is supplied with a binary operation corresponding to a mapping $\mu : K^2 \to K$, then K is called a groupoid.

If a binary operation μ is associative $(ab)c = a(bc)$ for every $a, b, c \in K$, then K is called a semigroup, where $\mu(a, b)$ is denoted shortly by ab.

An element $e = e_\mu$ in a groupoid K is called neutral (or unit), if $eb = be = b$ for each $b \in K$. A semigroup with a neutral element is called a monoid.

An element $b \in K$ in a groupoid with a unit is called left or right invertible, if there exists a left $b_l^{-1} \in K$ or right inverse $b_r^{-1} \in K$ respectively, i.e. $b_l^{-1}b = e$ or $bb_r^{-1} = e$ correspondingly. If an element is both left and right invertible, then it is called invertible.

A semigroup with a neutral element in which each element is invertible is called a group.

A groupoid with a neutral element in which each equation $ax = b$ or $xa = b$ has a solution is called a quasigroup.

Let K be a set and let two operations $+ : K^2 \to K$ the addition and $\times : K^2 \to K$ the multiplication be given so that K and $K \setminus \{0\}$ are monoids (or quasigroups or groupoids with with neutral elements) relative to $+$ and $\times$ correspondingly with neutral elements denoted by $e_+ =: 0$ and $e_\times =: 1$ so that $a \times 0 = 0 \times a = 0$ for each $a \in K$ and either the left distributivity $a(b+c) = ab + ac$ for every $a, b, c \in K$ or the right distributivity $(b + c)a = ba + ca$ for every $a, b, c \in K$ is accomplished, then K is called a semiring (or a quasiring respectively) with either the left or right distributivity correspondingly. If it is simultaneously right and left distributive, then it is called simply a distributive semiring (or a distributive quasiring respectively). If a type of the distributivity is not mentioned it will be supposed that the left or right

distributivity is accomplished in a semiring or a quasiring.

A semiring (or a quasiring) K, which is a group relative to the addition and $K\setminus\{0\}$ is a group (or a quasigroup) relative to the multiplication, is called a ring (or a non-associative ring respectively, i.e. non-associative relative to the multiplication).

A semiring K (or a quasiring, or a ring, or a non-associative ring) having also a structure of a linear space over a field $\mathbf{F}$ and such that $\alpha(a+b) = \alpha a + \alpha b$, $1a = a$, $\alpha(ab) = (\alpha a)b = a(\alpha b)$ and $(\alpha\beta)a = \alpha(\beta a)$ for each $\alpha, \beta \in \mathbf{F}$ and $a, b \in K$ is called a semialgebra (or a quasialgebra, or and algebra or a non-associative algebra correspondingly).

A set K with binary operations $\mu_1, ..., \mu_n$ will also be called an algebraic object. An algebraic object is commutative relative to an operation μ_p if $\mu_p(a, b) = \mu_p(b, a)$ for each $a, b \in K$.

A set K is called directed by a relation $\le$, if it satisfies the following conditions:

$(D1)$ if $x \le y$ and $y \le z$, then $x \le z$;

$(D2)$ for each $x \in K$ one has $x \le x$;

$(D3)$ for each $x, y \in K$ an element $z \in K$ exists so that $x \le z$ and $y \le z$.

If $x \le y$ and $x \ne y$, then one traditionally writes $x < y$.

If a relation $<$ on K satisfies the following conditions:

$(LO1)$ if $x < y$ and $y < z$, then $x < z$;

$(LO2)$ if $x < y$, then $y < x$ takes no place;

$(LO3)$ if $x \ne y$, then either $x < y$ or $y < x$,

then K is called linearly ordered, $<$ is a linear ordering on K.

(WO). If a set K is linearly ordered and each non-void subset A has a least element in K (i.e. K-least element), then K is called well-ordered.

An algebraic object K with binary operations $\mu_1, ..., \mu_n$ is called either directed or linearly ordered or well-ordered if it is such as a set correspondingly and its binary operations preserve an ordering: $\mu_p(a, b) \le \mu_p(c, d)$ for each $p = 1, ..., n$ and for every $a, b, c, d \in K$ so that $a \le c$ and $b \le d$ when a, b, c, d belong to the same linearly ordered set Z in K.

Henceforward, we suppose that the minimal element in an ordered K is zero.

Henceforth, for semialgebras, non-associative algebras or quasialgebras A speaking about ordering on them we mean that only their non-negative cones $K = \{y : \ y \in A, 0 \le y\}$ are considered. For non-negative cones K in semialgebras, non-associative algebras or quasialgebras only the case over the real field will be considered, since it is the unique topologically complete field in which the addition and multiplication operations are compatible with the natural linear ordering, i.e. K is over $[0, \infty) = \{x : \ 0 \le x, x \in \mathbf{R}\}$.

2. Definition. Suppose that a family $\{K_j : \ j \in J\}$ of algebraic objects K_j is given with binary operations $\mu_1, ..., \mu_n$, where J is a directed infinite set, so that algebraic embeddings

(1) $t^j_k : K_j \hookrightarrow K_k$ exist for each $k < j$ in J with

(2) $\mu_l(t^j_k(a), t^j_k(b)) = t^j_k(\mu_l(a, b))$ for all $a, b \in K_j$ and each $l = 1, ..., n$;

(3) $\psi_p : J \to J$ is a surjective bijective monotone increasing mapping so that $\psi_p(j) \le j$ and $\psi_p(k) < \psi_p(j)$ for each $k < j \in J$ for all $p = 1, ..., n$;

(4) $\phi_p : J \to J$ is a bijective monotone increasing mapping so that $j \le \phi_p(j)$ and $\phi_p(k) < \phi_p(j)$ for each $k < j \in J$ and for all $p = 1, ..., n$. We define a new algebraic object

(5) $s(K_j : \ j \in J)$ elements of which are $y = (y_j : \ y_j \in K_j \ \forall j \in J) \in \prod_{j\in J} K_j$ supplied with binary operations

(6) $\mu_p(y, z) = q$ such that $q_{\psi_p(j)} = \mu_p(y_j, t_j^{\phi_p(j)}(z_{\phi_p(j)}))$ for each $j \in J$, for all $y, z \in s(K_j : \ j \in J)$ and each $p = 1, ..., n$.

3. Proposition. *Let $\{K_j : \ j \in J\}$ be a family of algebraic objects with operations $\mu_1, ..., \mu_n$ and let an algebraic object $s(K_j : \ j \in J)$ be as in Definition 2. If each K_j is directed so that algebraic embeddings $t^j_k : K_j \hookrightarrow K_k$ are monotone increasing*

(1) *$t^j_k(a) < t^j_k(b)$ for each $a < b \in K_j$ and all $k < j \in J$*

and J is directed, then $s(K_j : \ j \in J)$ is naturally directed.

Proof. The algebraic object $s(K_j : \ j \in J)$ can be partially ordered:

(2) $y \le z \in s(K_j : \ j \in J)$ if and only if $y_j \le z_j \in K_j$ for each $j \in J$. Each algebraic object K_j is directed, hence an element $u \in s(K_j : \ j \in J)$ with $y_j \le u_j$ and $z_j \le u_j$ for each $j \in J$ satisfies according to Condition (2) the inequalities $y \le u$ and $z \le u$, consequently, the set $s(K_j : \ j \in J)$ is directed.

If $a \le c$ and $b \le d$ in $s(K_j : \ j \in J)$, then

(3) $\mu_p(a_k, t_k^{\phi_p(k)}(b_{\phi_p(k)})) \le \mu_p(c_k, t_k^{\phi_p(k)}(d_{\phi_p(k)}))$ for each $k \in J$ and each $p = 1, ..., n$, since $a_k \le c_k$ and $b_{\phi_p(k)} \le d_{\phi_p(k)}$. Thus $\mu_p(a, b) \le \mu_p(c, d)$ for each $p = 1, ..., n$ and hence the algebraic object $s(K_j : \ j \in J)$ is directed.

4. Proposition. *If $\{K_j : \ j \in J\}$ is a family of algebraic objects with operations $\mu_1, ..., \mu_n$ and an algebraic object $s(K_j : \ j \in J)$ is as in Definition 2 and if either each K_j is*

(1) *right $\mu_k(\mu_l(b, c), a) = \mu_l(\mu_k(b, a), \mu_k(c, a))$ distributive or*

(2) *left $\mu_k(a, \mu_l(b, c)) = \mu_l(\mu_k(a, b), \mu_k(a, c))$ distributive for each $a, b, c \in K_j$ for a pair of binary operations (μ_k, μ_l) with $k \ne l$, where $\psi_l = id$ and $\phi_l = id$, then $s(K_j : \ j \in J)$ is right or left distributive for (μ_k, μ_l) respectively.*

Proof. Since the mappings $\psi_l = id$ and $\phi_l = id$ are the identities on J, $id(j) = j$ for each $j \in J$, then for arbitrary elements $a, b, c \in s(K_j : \ j \in J)$ in the case (1) we get

$\mu_k(\mu_l(b_j, c_j), t_j^{\phi_k(j)}(a_{\phi_k(j)})) = \mu_l(\mu_k(b_j, t_j^{\phi_k(j)}(a_{\phi_k(j)})), \mu_k(c_j, t_j^{\phi_k(j)}(a_{\phi_k(j)})))$, consequently, $\mu_k(\mu_l(b, c), a) = \mu_l(\mu_k(b, a), \mu_k(c, a))$. Analogously in the case (2) the equalities are satisfied:

$\mu_k(a_j, \mu_l(t_j^{\phi_k(j)}(b_{\phi_k(j)}), t_j^{\phi_k(j)}(c_{\phi_k(j)}))) = \mu_l(\mu_k(a_j, t_j^{\phi_k(j)}(b_{\phi_k(j)})), \mu_k(a_j, t_j^{\phi_k(j)}(c_{\phi_k(j)})))$ and hence $\mu_k(a, \mu_l(b, c)) = \mu_l(\mu_k(a, b), \mu_k(a, c))$.

5. Theorem. *If each K_j is non-trivial (contains not only neutral elements relative to binary operations $\mu_1, ..., \mu_n$) and a number p exists, $1 \le p \le n$, such that $j < \phi_p(j)$ for each $j \in J$, then an algebraic object $s(K_j : \ j \in J)$ from Definition 2 is non-associative relative to a binary operation μ_p.*

Proof. For arbitrary three elements $a, b, c \in s(K_j : \ j \in J)$ we deduce from Definition 2:

(1) $\mu_p(\mu_p(a,b),c)) = q$ with

(2) $q_{\psi_p(\psi_p(j))} = \mu_p(\mu_p(a_j, t_j^{\phi_p(j)}(b_{\phi_p(j)})), t_{\psi_p(j)}^{\phi_p(\psi_p(j))}(c_{\phi_p(\psi_p(j))}))) \in K_{\psi_p(\psi_p(j))}$ and

(3) $\mu_p(a, \mu_p(b,c)) = v$ with

(4) $v_{\psi_p(\psi_p(j))} = \mu_p(a_{\psi_p(j)}, \mu_p(t_{\psi_p(j)}^{\phi_p(\psi_p(j))}(b_{\phi_p(\psi_p(j))}), t_{\phi_p(\psi_p(j))}^{\phi_p(\phi_p(\psi_p(j)))}(c_{\phi_p(\phi_p(\psi_p(j)))}))) \in K_{\psi_p(\psi_p(j))}$.

The associativity of μ_p, i.e. the equality $q = v$ is equivalent to

(5) $[\forall j \in J \ \ q_{\psi_p(\psi_p(j))} = v_{\psi_p(\psi_p(j))}]$.

Elements a, b, c are arbitrary and generally non-equal and independent. Therefore, Formulas $(1-5)$ imply that $a_j = a_{\psi_p(j)}$ for each $j \in J$, consequently, $\psi_p(j) = j$ for each $j \in J$. Then we get $c_{\phi_p(j)} = c_{\phi_p(\phi_p(j))}$ for each $j \in J$, but $k < \phi_p(k)$ for each $k \in J$ by the conditions of this theorem and generally there exists $c_l \ne c_k$ for $l \ne k \in J$, in particular, with $l = \phi_p(j)$ and $k = \phi_p(\phi_p(j))$. Thus Condition (5) is not satisfied, consequently, the algebraic object $s(K_j : \ j \in J)$ is non-associative relative to the binary operation μ_p.

6. Theorem. *If $\{X_k : \ k \in K\}$ is a family of non-associative non-trivial algebraic objects X_k with binary operations $\mu_1, ..., \mu_n$, K is a set, then there exists a non-associative algebraic abject K with binary operations $\mu_1, ..., \mu_n$ so that K is not isomorphic with X_k for each $k \in K$ even up to a direct sum decomposition with either some associative algebraic objects or the product X_k^m, $m \in \mathbf{N}$, with binary operations $\mu_1, ..., \mu_n$.*

Proof. For the set $P := \bigcup_{k \in K} X_k$ one can take a set M such that $|P| < |M|$ and $\aleph_0 < |M|$, where $|M|$ denotes the cardinality of M. Therefore, the inequality $|P| < |H \times M|$ is satisfied, where $H = \bigcup_{l \in \mathbf{N}} K^l$, $H \times M$ denotes the cartesian product of the sets H and M, K^l denotes the l-fold cartesian product of K with itself.

Let H be directed by inclusion $u \le v$ if and only if $u \subset v$ for any $u, v \in H$. Each set can be well-ordered in accordance with Zermelo's theorem. Therefore, let M be well-ordered. The cartesian product $J := H \times M$ can be directed lexicographically: $(u,l) < (v,k)$ if and only if either $u < v$ or $u = v$ and $l < k$, where $l, k \in M$ and $u, v \in H$. Without loss of generality we can suppose that J has not a maximal element. Then we put $K_j = \bigoplus_{i \in u} X_i$ for each $j = (u,l) \in J$ and take the algebraic object $s(K_j : \ j \in J)$. Each algebraic object X_k is non-trivial by the conditions of this theorem. We take ϕ_p on J such that $j < \phi_p(j)$ for each $j \in J$ for all $p = 1, ..., n$. The cardinality of $s(K_j : \ j \in J)$ is greater than the cardinality of each X_k, consequently, $s(K_j : \ j \in J)$ is not isomorphic to X_k for each $k \in K$.

On the other hand, $s(K_j : \ j \in J)$ is non-associative relative to each operation $\mu_1, ..., \mu_n$ by Theorem 5. In view of Formulas $5(1-5)$ even if $s(K_j : \ j \in J)$ contains either some associative sub-object A or the product $A = X_k^m$ having a direct sum complement B so that $s(K_j : \ j \in J) = A \oplus B$,

then $|A| \le |B|$ and hence $|B| = |s(K_j : \ j \in J)|$, since $\aleph_0 < |M| \le |H \times M|$. Therefore, the inequality $|X_k| < |B|$ is satisfied for each $k \in K$, hence B is not isomorphic with X_k for each $k \in K$.

7. Proposition. *If $\{K_j : \ j \in J\}$ is a family of linearly or well-ordered algebraic objects with binary operations $\mu_1, ..., \mu_n$ preserving ordering, i.e. $\mu_p(a,b) < \mu_p(c,d)$ for each $p = 1,...,n$ when either $a < c$ and $b \le d$ or $a \le c$ and $b < d$, where $J \subset \mathbf{Z}$ is a countable set, then their product $K = \prod_{j \in J} K_j$ is the algebraic object with binary operations $\mu_1, .., \mu_n$ and it can be naturally well-ordered.*

Proof. The product K can be supplied with the binary operations

(1) $\mu_p(a,b) = (\mu_p(a_j, b_j) \in K_j : \ j \in J)$ for each $p = 1,...,n$, for every $a, b \in K, \ a = (a_j : \ \forall j \ a_j \in K_j)$.

By the Zermelo's theorem a set J can be well-ordered. Since J is well-ordered, there exists a J-least element $j = \inf B_{y,z}$ of the subset $B_{y,z} := \{l : \ l \in J, \ y_l \ne z_l\}$, where $y, z \in s(K_j : \ j \in J)$ are two arbitrary chosen elements, $y = (y_l : \ \ \forall l \in J \ y_l \in K_l)$. On the other hand, $J \subset \mathbf{Z}$ by the conditions of this proposition, consequently, $j \in B_{y,z}$. There exists the lexicographic linear ordering:

(2) $y < z \in s(K_j : \ \ j \in J)$ if there exists $j(y,z) = j \in J$ such that $y_k = z_k$ for each $k < j$, whilst $y_j < z_j$. As usually we put $y \le z$ if $y = z$ or $y < z$. The set K is well-ordered, if K_j is well-ordered for each $j \in J$, since if $A \subset K$ and $B = \{k \in J : \ \exists y, z \in A; \ y_k \ne z_k\}$, then $j = \inf B \in J$ and $\inf A_j \in K_j$, where $A_j = \{y_j \in K_j : \ y \in A\}$.

Let either $a < c$ and $b = d$ or $a = c$ and $b < d$, then from Formulas $(1,2)$ and the conditions of this proposition it follows that $\mu_p(a,b) < \mu_p(c,d)$ in K. If $a < c$ and $b < d$ and $j(a,c) \le j(b,d)$, then $a_l < c_l$ and $b_l \le d_l$ for $l = j(a,c)$, consequently, $\mu_p(a_l, b_l) < \mu_p(c_l, d_l)$. Analogously, if $a < c$ and $b < d$ and $j(b,d) \le j(a,c)$, then $a_l \le c_l$ and $b_l < d_l$ for $l = j(b,d)$, consequently, $\mu_p(a_l, b_l) < \mu_p(c_l, d_l)$. Thus the binary operation μ_p preserves the ordering on K for each $p = 1,...,n$.

8. Definition. A subalgebra A (or a subquasiring, or a subsemiring) of an (a non-associative) algebra K (or a quasiring, or a semiring) is called a left or right ideal if $AK \subseteq A$ or $KA \subseteq A$ respectively, where $AB := \{c \in K : \ c = ab, a \in A, b \in B\}$. If A is a left and right ideal simultaneously, then A is called an ideal.

An (a non-associative) algebra (or a quasiring, or a semiring) is called simple if it does not contain ideals different from $\{0\}$ and K, where K is non-trivial, $K \ne \{0\}$.

9. Definitions. Let $s_a(K_j : \ \ j \in J)$ denote the subalgebraic object consisting of all elements $y \in s(K_j : \ \ j \in J)$ (see Definition 2) such that $y = (y_j : \ \forall j \in J \ y_j \in K_j)$ with a finite set $J_y := \{j : \ \ j \in J, y_j \ne 0\}$, where $0 = e_{\mu_1, j} \in K_j$ is a neutral element in K_j relative to a binary operation μ_1 for each j.

One says that a quasiring or an algebraic object K or a non-associative algebra over a field $\mathbf{F}$ is not finitely (countably) generated, if for each finite

(countable respectively) family of its elements $a_1, a_2, ... \in K$ a minimal subquasiring or a minimal subalgebraic object Z in K or a subalgebra Z over the same field $\mathbf{F}$ correspondingly containing these elements, $a_1, a_2, ... \in Z$, does not coincide with K, $Z \ne K$.

10. Theorem. *If X is a simple non-associative algebra or a quasiring so that X is not finitely generated, then there exist a simple non-associative algebra or a quasiring K correspondingly and an embedding $X \hookrightarrow K$ so that X is not isomorphic with K.*

Proof. We take any infinite set I such that $\aleph_0 \le |I| < |2^X|\aleph_0$. Without loss of generality we suppose that a set I is well-ordered, since by Zorn's theorem it can be well-ordered. Let X contain a neutral element 0 by the addition $+ = \mu_1$. Otherwise one can enlarge X supplying it with a neutral element. Therefore, without loss of generality $0 \in X$. Then we choose a subset $J \subset I$ without a maximal element (and without a minimal element if necessary) so that $|J| = |I|$. Then we consider $K := s_a(X_j : \ j \in J)$, where $X_j = X$ for each $j \in J$ (see Definition 9). For an addition operation μ_1 we take $\psi_1(j) = j$ and $\phi_1(j) = j$ for each $j \in J$, while for a multiplication operation μ_2 we choose $\psi_2(j) \le j$ and $j < \phi_2(j)$ for each $j \in J$ (see Definition 2). Since $X_j = X$ for each $j \in J$ we put $t^j_k = id$ for every $k \le j \in J$.

For each natural number n any family of elements $a_1, ..., a_n$ does not generate X as either the non-associative algebra or the quasiring or the semiring respectively, since X is not finitely generated.

In accordance with Theorem 5 the algebraic object K is non-associative relative to the multiplication μ_2. By Proposition 4 K is a non-associative algebra or a quasiring correspondingly.

From Definition 2 it follows that $\theta(\mu_p(x, z)) = \mu_p(\theta(x), \theta(z))$ for each $x, z \in X$ for $p = 1$ and $p = 2$, where

(1) $\theta(x) = (x_j : \ x_j = x \forall j \in J)$, $\theta : X \hookrightarrow K$. On the other hand, if X is a non-associative algebra, then $\theta(\alpha x) = (y_j : \ \forall j \in J \ \ y_j = \alpha x) = \alpha\theta(x)$ for each $\alpha \in \mathbf{F}$ and $x \in X$. Thus θ is the embedding of the non-associative algebra or a quasiring X into the non-associative algebra or a quasiring K correspondingly.

Let M be a non-trivial ideal in K, that is $M \ne \{0\}$. We take a family

(2) T of all elements $y \in K$ of the form $y = (y_j : \ y_j = x\delta_{k,j}, \ j \in J, k \in P)$, where $x \in X$, $\ \delta_{k,j} = 0$ for each $k \ne j$, while $\delta_{j,j} = 1$, $\ P$ is a finite subset in J, particularly, P may be a singleton. An ideal M is a subalgebra or a subquasiring correspondingly in K. From the inclusions $MK \subseteq M$ and $KM \subseteq M$ it follows that the products $\mu_2(y, a)$ and $\mu_2(a, y)$ are in M for each $a \in M$ and $y \in T$. Therefore from formulas for μ_1 and μ_2 on K it follows that M contains T. Thus, the inclusion

(3) $\theta(X) \subset M$ is fulfilled.

The non-associative algebra or the quasiring $\theta(X)$ is simple and from Conditions $(1-3)$ it follows by multiplications μ_2 and additions μ_1, that for each finite subset P in J and every $x_j \in X$ for each $j \in P$ there exists an element $z \in M$ so that $z_j = x_j$ for each $j \in P$, consequently, $M = K$. Thus

the non-associative algebra or the quasiring K correspondingly is simple.

11. Theorem. *Let $\{X_i :\ i \in H\}$ be a family of simple pairwise non-isomorphic either non-associative algebras or quasirings and let each X_i be not finitely generated, where H is an infinite linearly ordered set and without a maximal element. Let also for each $i < k$ an embedding $u_i^k : X_k \hookrightarrow X_i$ exist. Then there is a simple non-associative algebra or a quasiring K correspondingly so that X_i is not isomorphic with K for each $i \in H$.*

Proof. For each X_i consider a set J_i without a maximal element and a minimal element as J for X in Theorem 9. Then we take the set $J = \bigcup_{i\in H}(i, J_i)$ and order it lexicographically: $(i, j) < (k, l)$ if either $i < k \in H$ or $i = k$ and $j < l \in J_i$. Let $K_{i,j} = X_i$ for each $j \in J_i$ and let $K = s_a(K_{i,j} :\ (i, j) \in J)$.

We then choose an addition operation μ_1 with $\psi_1(i, j) = (i, j)$ and $\phi_1(i, j) = (i, j)$ for each $(i, j) \in J$, while for a multiplication operation μ_2 let $\psi_2(i, j) = (l, m)$ with $l \le i$ and $m \le j$, whilst $\phi_2(i, j) = (v, w)$ with $i < v$ and $j < w$ for each $(i, j) \in J$. We put $t_{(i,j)}^{(k,w)} = u_i^k$ for every $(i, j) \le (k, w) \in J$, where $u_i^i = id$. The rest of the proof is analogous to that of Theorem 9. It shows that K is simple and K is not isomorphic with X_i as a non-associative algebra or a quasiring respectively for each $i \in H$.

12. Corollary. *A family of all simple either non-associative algebras $\mathcal{F}_a$ or quasirings $\mathcal{F}_q$, which are not pairwise isomorphic, is not a set, i.e. $\mathcal{F}_a$ and $\mathcal{F}_q$ are proper classes (in the NBG axiomatic).*

Proof. This follows from Theorems 5, 6, 10 and 11. Indeed, if K is either a non-associative algebra or a quasiring, then either a quotient (non-associative) algebra or a quotient (quasi) ring K/A by its ideal A is an algebra or a (quasi) ring respectively. If A is a maximal ideal, then K/A is simple. Even if K/A is associative, there exists a non-associative algebra or a quasiring X generated by K/A according to Theorem 5. In view of Theorem 10 it is simple and either a non-associative algebra or a quasiring correspondingly. In view of Theorems 5 and 10 applied by induction there exists a sequence of simple non-associative algebras or quasirings respectively satisfying conditions of Theorem 11, since I and hence J can be taken uncountable for non countably generated X. On the other hand, for a family of simple either non-associative algebras or quasirings correspondingly linearly ordered by algebraic embeddings a non-associative algebra or a quasiring respectively non-isomorphic to any of them exists. That is, the family of all simple either non-associative algebras or qiasirings is infinite.

If T is a set, it can be well-ordered by Zorn's theorem. Taking a set J in the class On of all ordinals so that J has the same cardinality as an infinite set T and without a maximal element, one gets that an infinite set P can be linearly ordered as J without a maximal element. Therefore, if a family $\{K_j :\ j \in J\}$ of pairwise non-isomorphic non-associative algebras or quasirings is taken with an infinite set J, then either a non-associative algebra or a quasiring X correspondingly non-isomorphic to each K_j would exist.

Proposition 4.7(1) [10] states that $|-Ord(X) \supset (X \notin X \& \forall u (u \in X \supset u \notin u))$, while Proposition 4.7(9) [10] asserts: $|-Ord(X) \supset X = On \vee X \in On$. This gives the contradiction $J \in J$, since $J \in On$. Therefore, P does not belong to any class and hence is not the set (see the definition in §4.1 [10]). Thus P is not the set in the NBG axiomatic. That is $\mathcal{F}_a$ and $\mathcal{F}_q$ are proper classes for a family of all pairwise non-isomorphic non-associative algebras or quasirings respectively.

13. Remark. Corollary 12 is consistent with the fact that the class On of all ordinals is not a set in the NBG theory (see Proposition 4.7(8) in [10]). Thus constructions given above show that there are many different directed or linearly ordered or well ordered non-associative algebras and quasirings.

The class On of all ordinals has the addition $\mu_1 = +_o$ and the multiplication $\mu_2 = \times_o$ operations which are generally non-commutative, associative, with unit elements 0 and 1 respectively, on On the right distributivity is satisfied (see Propositions 4.29-4.31 and Examples 1-3 in [10]).

For each non-void set A in On there exists $\sup A \in On$ (see [6]).

If K is a linearly ordered non-commutative relative to the addition semiring (or a quasiring), then the new operation $(a,b) \mapsto \max(a,b) =: a \bigoplus b$ defines the commutative addition. Then $c(a \bigoplus b) = \max(ca, cb) = ca \bigoplus cb$ and $(a \bigoplus b)c = \max(ac, bc) = ac \bigoplus bc$ for every $a, b, c \in K$, that is $(T, \bigoplus, \times)$ is left and right distributive.

14. Definitions. A partial order is a pair $< T, \le >$ such that $T \ne \emptyset$ and $\le$ is a transitive and reflexive relation on T. A pair $< T, \le >$ is a partial order in the strict sense if and only if it in addition satisfies $\forall p, q (p \le q \wedge q \le p \to p = q)$. Then one defines $p < q$ if and only if $p \le q$ and $p \ne q$.

A tree is a partial order in the strict sense, such that for each $x \in T$ the set $\{y \in T : y < x\}$ is well-ordered.

We consider a directed set K which satisfies the condition:

(DW) for each linearly ordered subset A in K there exists a well-ordered subset B in K such that $A \subset B$.

15. Corollary. *Let $\{K_j : \ j \in J\}$ be a family of directed algebraic objects satisfying the condition (DW) with binary operations $\mu_1, ..., \mu_n$ so that algebraic embeddings $t^j_k : K_j \hookrightarrow K_k$ are monotone increasing*

(1) $t^j_k(a) < t^j_k(b)$ *for each* $a < b \in K_j$ *and all* $k < j \in J$,

where J is a directed set. Then the algebraic object $s(K_j : \ j \in J)$ (see Definition 2) can be directed to satisfy the condition (DW) as well.

Proof. We consider the algebraic object $s(K_j : \ j \in J)$ directed as in Proposition 3: $x \le y \in s(K_j : \ j \in J)$ if and only if $x_j \le y_j$ for each $j \in J$. Therefore, if $x \le y$ and $x \ne y$, then there exists $j \in J$ such that $x_j \ne y_j$, consequently, $x_j < y_j$. We put

(2) $x < y$ if and only if $x \le y$ and $x \ne y$.

There are natural projections $\pi_j : s(K_j : \ j \in J) \to K_j$ so that $\pi_j(y) = y_j$ for each $j \in J$. For a linearly ordered subset A in $s(K_j : \ j \in J) \to K_j$ for each projection $\pi_j(A) = A_j$ one can take a family of all well-ordered subsets $B_{j,v} \in K_j$ so that $A_j \subset B_{j,v}$ for each $v \in V_j$, where V_j is a set. Then each set

$B_j := \cap_{v\in V_j} B_{j,v}$ is well-ordered and contains A_j. Let $C = \prod_{j\in J} B_j$. Suppose that $E \subset A$, then $g = (g_j : \ \forall j \in J \ g_j = \inf E_j)$ is in C, since $g_j \in B_j$ for each $j \in J$. Therefore, $g \le \inf E$, i.e. $\inf E =: u$ is in C, but then there exists $y \in A$ so that $u \le y$. Put

(3) $B = A \cup (\bigcup_{E\subset A} \inf E)$.

If $U \subset B$, then there exist sets H and F_b for each $b \in H$ so that

$U = (B \cap A) \cup (\bigcup_{b\in H} \inf F_b)$, consequently,

(4) $\inf U = \inf((B \cap A) \cup (\bigcup_{b\in H} F_b)) \in B$.

On the other hand, if $f, h \in B$, then

(5) $f = \inf F_f$ and $h = \inf F_h$ for some suitable subsets F_f and F_h in A. Suppose that these sets F_h are chosen to satisfy Conditions $(4, 5)$. There may be several variants. If $F_f \subset F_h$, then $h \le f$; if $F_h \subset F_f$, then $f \le h$. If $F_f \setminus F_h$ contains and element y so that $y < c$ for each $c \in F_h$, then $f \le h$. If $F_h \setminus F_f$ contains an element y so that $y < c$ for each $c \in F_f$, then $h \le f$. If for each $a \in F_h$ there exists $c \in F_f$ such that $c \le a$ and for each $e \in F_f$ there exists $q \in F_h$ such that $q \le e$, then $f = h$. Together with (2) this gives that either $f < h$ or $h < f$ or $f = h$, i.e. B is linearly ordered and together with (4) this implies that B is well-ordered and (3) means that $A \subset B$. Thus the algebraic object $s(K_j : \ j \in J)$ is directed so that it satisfies the condition (DW).

3 Skew idempotent functionals

1. Definitions. Let K be a well-ordered or directed satisfying condition 2.14(DW) either semiring or quasiring (or a non-negative cone in a quasialgebra over the real field $\mathbf{R}$) such that

(1) $\sup E \in K$ for each $E \in T$, where T is a family of subsets of K.

For a set X and a semiring (or quasiring) K let $C(X, K)$ denote a semiring (or a quasiring respectively) of all mappings $f : X \to K$ with the point-wise addition $(f + g)(x) = f(x) + g(x)$ and the point-wise multiplication $(fg)(x) = f(x)g(x)$ operations for every $f, g \in C(X, K)$ and $x \in X$.

If K is a directed semiring (or a directed quasiring) and X is a linearly ordered set, $C_+(X, K)$ (or $C_-(X, K)$) will denote the set of all monotone non-decreasing (or non-increasing correspondingly) maps $f \in C(X, K)$.

For the space $C(X, K)$ (or $C_+(X, K)$ or $C_-(X, K)$) we suppose that

(2) a family T of subsets of K contains the family $\{f(X) : \ f \in C(X, K)\}$ (or $\{f(X) : \ f \in C_+(X, K)\}$ or $\{f(X) : \ f \in C_-(X, K)\}$ correspondingly) and K satisfies Condition (1).

Henceforward, we suppose that the minimal element in K is zero.

If a non-associative algebra or a quasialgebra A over $\mathbf{R}$ is considered, we suppose that K is a cone of non-negative elements (non-negative cone) in it (see also §2.1).

2. Remark. As an example of a semiring (or a quasiring) K in Definitions 1 one can take $K = On$ or $K = \{A : \ A \in On, |A| \le b\}$, where b

is a cardinal number such that $\aleph_0 \le b$ (see also Remark 2.13). Evidently, $K = On$ satisfies Condition 1(1), since $\sup E$ exists for each set E in On (see [6]).

Another examples are provided by Theorems 2.5, 2.10, 2.11, Propositions 2.3, 2.4, 2.7 and Corollaries 2.12, 2.15.

It is possible to modify Definition 1 in the following manner. For a well-ordered K without Condition 1(1) one can take the family of all bounded functions $f : X \to K$ and denote this family of functions by $C(X, K)$ for the uniformity of the notation.

For a directed K satisfying Condition 2.14(DW) without Condition 1(1) it is possible to take the family of all monotone non-decreasing (or non-increasing) bounded functions $f : X \to K$ for a linearly ordered set X and denote this family by $C_+(X, K)$ (or $C_-(X, K)$ correspondingly) also.

Naturally, $C(X, K)$ has also the structure of the left and right module over the semiring (or the quasiring correspondingly) K, i.e. af and fa belong to $C(X, K)$ for each $a \in K$ and $f \in C(X, K)$. To any element $a \in K$ the constant mapping $g^a \in C(X, K)$ corresponds such that $g^a(x) = a$ for each $x \in X$. If K is right (or left) distributive, then $q(f + h) = qf + qh$ (or $(f + h)q = fq + hq$ correspondingly) for every $q, f, h \in C(X, K)$.

The semiring (or the quasiring) $C(X, K)$ will be considered directed:

(1) $f \le g$ if and only if $f(x) \le g(x)$ for each $x \in X$.

Indeed, if $f, h \in C(X, K)$, then $a = \sup(f(X)) \in K$ and $b = \sup(h(X)) \in K$ according to Condition 1(1). Then there exists $c \in K$ so that $a \le c$ and $b \le c$, consequently, $f \le g^c$ and $h \le g^c$. Thus for each $f, h \in C(X, K)$ a function $q \in C(X, K)$ exists so that $f \le q$ and $h \le q$. From $a + b \le c + d$ and $ac \le bd$ for each $a \le c$ and $b \le d$ in K it follows that $f + q \le g + h$ and $fq \le gh$ for each $f \le g$ and $q \le h$ in $C(X, K)$.

If $f \le g$ and $f \ne g$ (i.e. $\exists x \in X$ $f(x) \ne g(x)$), then we put $f < g$.

For a mapping $f \in C(X, K)$ its support $supp(f)$ is defined as usually

(2) $supp(f) := \{x : \ x \in X, f(x) \ne 0\}$.

3. Lemma. *If E is a subset in X, then $C(X, K|E) := \{f : \ f \in C(X, K), supp(f) \subset E\}$ is an ideal in $C(X, K)$.*

Proof. If $f \in C(X, K|E)$ and $g \in C(X, K)$, then $f(x)g(x) = 0$ and $g(x)f(x) = 0$ when $f(x) = 0$, consequently, $supp(fg)$ and $supp(gf)$ are contained in E. Moreover, if $f, h \in C(X, K|E)$, then $supp(f+h)$ and $supp(h+f)$ are contained in E, since $f(x) + h(x) = 0$ and $h(x) + f(x) = 0$ for each $x \in X \setminus E$. Thus $C(X, K|E)$ is a semiring (or a quasiring respectively) and $C(X, K|E)C(X, K) \subseteq C(X, K|E)$ and $C(X, K)C(X, K|E) \subseteq C(X, K|E)$.

4. Corollary. *If E is a subset in X, then $C(E, K)$ is an ideal in $C(X, K)$.*

Proof. For a subset E in X one gets $C(E, K)$ isomorphic with $C(X, K|E)$, since each $f \in C(E, K)$ has the zero extension on $X \setminus E$.

5. Lemma. *For a linearly ordered set X and a directed semiring (quasiring) K there are directed semirings (or quasirings correspondingly) $C_+(X, K)$ and $C_-(X, K)$.*

Proof. The sets $C_+(X,K)$ and $C_-(X,K)$ are directed according to Condition 2(1) with a partial ordering inherited from $C(X,K)$. Since $a+b\le c+d$ and $ac\le bd$ for each $a\le c$ and $b\le d$ in K, then $f+q\le g+h$ and $fq\le gh$ for each $f\le g$ and $q\le h$ in $C_+(X,K)$ and $C_-(X,K)$. On the other hand, for each $f,h\in C(X,K)$ there exists $g^c\in C(X,K)$ so that $f\le g^c$ and $h\le g^c$ (see §2). If $f(x)\le f(y)$ and $h(x)\le h(y)$ for $f,h\in C_+(X,K)$ and each $x\le y$ in X, then $f(x)+h(x)\le f(y)+h(y)$ and $f(x)h(x)\le f(y)h(y)$, consequently, $f+h$ and fh are in $C_+(X,K)$. Analogously, if $f,h\in C_-(X,K)$, then $f+h$ and fh are in $C_-(X,K)$. But a constant mapping g^c belongs to $C_+(X,K)$ and $C_-(X,K)$. Thus $C_+(X,K)$ and $C_-(X,K)$ are directed semirings (or quasirings correspondingly).

6. Lemma. *Suppose that K satisfies Conditions 2.14(DW) and 1$(1,2)$. Then the functions*

(1) $f\vee g(x):=\max(f(x),g(x))$ *and*

(2) $f\wedge g(x):=\min(f(x),g(x))$

are in $C(X,K)$ (or in $C_+(X,K)$ or in $C_-(X,K)$) for every pair of functions $f,g\in C(X,K)$ (or in $C_+(X,K)$ or in $C_-(X,K)$ correspondingly) satisfying the condition:

(3) *for each $x\in X$ either $f(x)<g(x)$ or $g(x)<f(x)$ or $f(x)=g(x)$.*

Proof. Let $f,g\in C(X,K)$ satisfy Condition (3). Then the sets $\{x:\ x\in X, f(x)\le g(x)\}$ and $\{x:\ x\in X, f(x)\le g(x)\}$ are contained in X. For each subset E in K the sets

$(f\vee g)^{-1}(E)=[f^{-1}(E)\cap\{x:\ x\in X, g(x)\le f(x)\}]\cup[g^{-1}(E)\cap\{x:\ x\in X, f(x)\le g(x)\}]$ and

$(f\wedge g)^{-1}(E)=[f^{-1}(E)\cap\{x:\ x\in X, f(x)\le g(x)\}]\cup[g^{-1}(E)\cap\{x:\ x\in X, g(x)\le f(x)\}]$

are contained in X, consequently, the mappings $f\vee g$ and $f\wedge g$ are in $C(X,K)$, since either $\sup f\le\sup g$ or $\sup g<\sup f$ for f and g satisfying (3) and hence $\max(\sup f,\sup g)\in K$ and $\min(\sup f,\sup g)\in K$.

If $f,g\in C_+(X,K)$ and $x<y\in X$, then $f(x)\le f(y)$ and $g(x)\le g(y)$. If $f(x)\le g(x)$ and $g(y)\le f(y)$, then $f\vee g(x)=g(x)\le g(y)\le f(y)=f\vee g(y)$ and $f\wedge g(x)=f(x)\le g(x)\le g(y)=f\wedge g(y)$. If $f(x)\le g(x)$ and $f(y)\le g(y)$, then $f\vee g(x)=g(x)\le g(y)=f\vee g(y)$ and $f\wedge g(x)=f(x)\le f(y)=f\wedge g(y)$. Therefore, $f\vee g(x)\le f\vee g(y)$ and $f\wedge g(x)\le f\wedge g(y)$. Thus $f\vee g$ and $f\wedge g\in C_+(X,K)$. Analogously if $f,g\in C_-(X,K)$, then $f\vee g$ and $f\wedge g\in C_-(X,K)$.

7. Notation. Let $\bigodot$ denote the mapping on $[K\times C(X,K)]\cup[C(X,K)\times K]$ with values in $C(X,K)$ such that

(1) $c\bigodot f=g^c+f$ and $f\bigodot c=f+g^c$ for each $c\in K$ and $f\in C(X,K)$, where $g^c(x)=c$ for each $x\in X$, whilst the sum is taken point-wise $(f+g)(x)=f(x)+g(x)$ for every $f,g\in C(X,K)$ and $x\in X$.

8. Definition. We call a mapping ν on $C(X,K)$ (or $C_+(X,K)$ or $C_-(X,K)$) with values in K an idempotent (K-valued) functional if it satisfies for each $f,g,g^c\in C(X,K)$ (or in $C_+(X,K)$ or $C_-(X,K)$ correspondingly) the following five conditions

(1) $\nu(g^c) = c$;
(2) $\nu(c \bigodot f) = c \bigodot \nu(f)$ and
(3) $\nu(f \bigodot c) = \nu(f) \bigodot c$;
(4) $\nu(f \vee g) = \nu(f) \vee \nu(g)$ when f, g satisfy Condition 6(3) and
(5) $\nu(f \wedge g) = \nu(f) \wedge \nu(g)$ if f, g satisfy Condition 6(3),
where $a \vee b = \max(a, b)$ and $a \wedge b = \min(a, b)$ for each $a, b \in K$ when either $a < b$ or $a = b$ or $b < a$.

A mapping (functional) ν on $C(X, K)$ (or $C_+(X, K)$ or $C_-(X, K)$) with values in K we call order preserving (non-decreasing), if

(6) $\nu(f) \le \nu(g)$ for each $f \le g$

in $C(X, K)$ (or $C_+(X, K)$ or $C_-(X, K)$ respectively), i.e. when $f(x) \le g(x)$ for each $x \in X$.

A functional ν is called left or right K-homogeneous on $C(X, K)$ (or $C_+(X, K)$ or $C_-(X, K)$) if

(7) $\nu(bf) = b\nu(f)$ or
(8) $\nu(fb) = \nu(f)b$

for each f in $C(X, K)$ (or $C_+(X, K)$ or $C_-(X, K)$ correspondingly) and $b \in K$. A functional left and right homogeneous simultaneously is called homogeneous.

9. Remark. If a functional satisfies Condition 10(4), then it is order preserving.

The Dirac functional δ_x defined by the formula:

(1) $\delta_x f = f(x)$

is the idempotent K-homogeneous functional on $C(X, K)$, where x is a marked point in X.

If functionals $\nu_1, ..., \nu_n$ are idempotent and the multiplication in K is distributive, then for each constants

(2) $c_1 > 0, ..., c_n > 0$ in K with
(3) $c_1 + ... + c_n = 1$ functionals
(4) $c_1\nu_1 + ... + c_n\nu_n$ and
(5) $\nu_1 c_1 + ... + \nu_n c_n$

are idempotent. Moreover, if the multiplication in K is associative and distributive and constants satisfy Conditions $(2, 3)$ and functionals $\nu_1, ..., \nu_n$ are K-homogeneous, then functionals of the form $(4, 5)$ are also K-homogeneous.

The considered here theory is different from the usual real field $\mathbf{R}$, since $\mathbf{R}$ has neither an infimum nor a supremum, i.e. it is not well-ordered and satisfy neither $2.14(DW)$ nor 1(1).

10. Lemma. *Suppose that either*

(1) *K is well-ordered and satisfies Conditions 1$(1, 2)$ or*

(2) *X is linearly ordered and K is directed and satisfies Conditions 2.14(DW) and 1$(1, 2)$. Then there exists an idempotent K-homogeneous functional ν on $C(X, K)$ in case (1), on $C_+(X, K)$ and $C_-(X, K)$ in case (2). Moreover, if $K \in On$ and K is infinite, $\aleph_0 \le |K|$, or $K = On$, X is not a singleton, $|X| > 1$, then ν has not the form either 9(4) or 9(5) with Dirac functionals $\nu_1, ..., \nu_n$ relative to the standard addition in On.*

Proof. Suppose that ν is an order preserving functional on $C(X,K)$ (or $C_+(X,K)$ or $C_-(X,K)$). If functions f,g in $C(X,K)$ (or $C_+(X,K)$ or $C_-(X,K)$ respectively) satisfy Condition 6(3), then in accordance with Lemma 6 there exists $f\vee g$ and $f\wedge g$ in the corresponding $C(X,K)$ (or $C_+(X,K)$ or $C_-(X,K)$). Since $f\vee g\geq f$ and $f\vee g\geq g$ and $f\wedge g\leq f$ and $f\wedge g\leq g$ and the functional ν is order preserving, then $\nu(f)\vee\nu(g)\leq\nu(f\vee g)$ and $\nu(f\wedge g)\leq\nu(f)\wedge\nu(g)$.

Let also E be a subset in X, we put

(3) $\nu(f)=\nu_E(f)=\sup_{x\in E}f(x)$.

This functional exists due Conditions $1(1,2)$, since in both cases (1) and (2) of this lemma, the image $f(E)$ is linearly ordered and is contained in K.

From the fact that the addition preserves ordering on K (see §2.1) it follows that Properties $(1-3,7,8)$ are satisfied for the functional ν given by Formula (3). If $f\leq g$ on X, then for each $a\in f(E)$ there exists $b\in g(E)$ so that $a\leq b$, consequently, $\nu(f)\leq\nu(g)$, i.e 8(6) is fulfilled.

We consider any pair of functions f,g in $C(X,K)$ (or $C_+(X,K)$ or $C_-(X,K)$) satisfying Condition 6(3). In case (2) a set X is linearly ordered, in case (1) K is well-ordered, hence $f(X)$, $g(X)$, $f(E)$ and $g(E)$ are linearly ordered in K. Then for each $a\in f(E)\cup g(E)$ there exist $b\in(f\vee g)(E)$ so that $a\leq b$, while for each $c\in(f\vee g)(E)$ there exists $d\in f(E)\cup g(E)$ so that $c\leq d$, hence $\nu(f\vee g)=\nu(f)\vee\nu(g)$. Moreover, for each $a\in f(E)\cup g(E)$ there exists $b\in(f\wedge g)(E)$ so that $b\leq a$ and for each $c\in(f\wedge g)(E)$ there exists $d\in f(E)\cup g(E)$ so that $d\leq c$, consequently, $\nu(f\wedge g)=\nu(f)\wedge\nu(g)$. Thus Properties $8(4,5)$ are satisfied as well.

If a set X is not a singleton, $|X|>1$, and $K\subset On$ is infinite, $\aleph_0\leq|K|$, then taking a set E in X different from a singleton, $|E|>1$, we get that the functional given by Formula (3) can not be presented with the help of Dirac functionals $\nu_1=\delta_{x_1},...,\nu_n=\delta_{x_n}$ by Formula either 9(4) or 9(5) relative to the standard addition in On, since functions f in $C(X,K)$ (or $C_+(X,K)$ or $C_-(X,K)$) separate points in X (see Remark 2).

11. Remark. Relative to the idempotent addition $x\vee y=\max(x,y)$ the functional ν_E given by 10(3) has the form $\nu_E(f)=\vee_{x\in E}\delta_x(f)$.

Let $I(X,K)$ denote the set of all idempotent K-valued functionals on $C(X,K)$, $I_l(X,K)$ of all idempotent K-valued functionals on $C_+(X,K)$, let also $I_h(X,K)$ and $I_{l,h}(X,K)$ denote their subsets of idempotent homogeneous functionals.

12. Definitions. A functional $\nu:C(X,K)\to K$ is called weakly additive, if

(1) $\nu(h+g^c)=\nu(h)+c$ and $\nu(g^c+h)=c+\nu(h)$ for all $c\in K$ and $h\in C(X,K)$;

(2) order preserving if $\nu(f)\leq\nu(h)$ for each $f\leq h\in C(X,K)$;

(3) normalized at $c\in K$, if $\nu(g^c)=c$;

(4) non-expanding if $\nu(f)\leq\nu(h)+c$ when $f\leq h+g^c$ and $\nu(f)\leq c+\nu(h)$ when $f\leq g^c+h$ for any $f,h\in C(X,K)$ and $c\in K$,

where ν may be non-linear as well.

The family of all order preserving weakly additive functionals on $C(X,K)$ (or $C_+(X,K)$) with values in K will be denoted $\mathcal{O}(X,K)$ (or $\mathcal{O}_l(X,K)$ respectively).

If $E \subset C(X,K)$ (or $E \subset C_+(X,K)$) satisfies the conditions: $g^0 \in E$, $g+b$ and $b+g \in E$ for each $g \in E$ and $b \in K$, then E is called an $\mathcal{A}$-subset.

13. Lemma. *If $\nu : C(X,K) \to K$ is an order preserving weakly additive functional, then it is non-expanding.*

Proof. Suppose that $f, h \in C(X,K)$ and $b \in K$ are such that $f(x) \le (h(x)+c)$ or $f(x) \le (c+h(x))$ for each $x \in X$, then $12(1,2)$ imply that $\nu(f) \le (\nu(h)+c)$ or $\nu(f) \le (c+\nu(h))$ respectively. Thus the functional ν is non-expanding.

14. Lemma. *Suppose that A is an $\mathcal{A}$-subset (or a left or right submodule over K) in $C(X,K)$ (or in $C_+(X,K)$)and $\nu : A \to K$ is an order preserving weakly additive functional (or left or right K-homogeneous with left or right distributive quasi-ring K correspondingly). Then there exists an order preserving weakly additive (or left or right K-homogeneous correspondingly) functional $\mu : C(X,K) \to K$ (or $\mu : C_+(X,K) \to K$ respectively) such that its restriction on A coincides with ν.*

Proof. One can consider the set $\mathcal{F}$ of all pairs (B,μ) so that B is an $\mathcal{A}$-subset (or a left or right submodule over K respectively), $A \subseteq B \subseteq C(X,K)$, μ is an order preserving weakly additive functional on B the restriction of which on A coincides with ν. The set $\mathcal{F}$ is partially ordered: $(B_1,\mu_1) \le (B_2,\mu_2)$ if $B_1 \subseteq B_2$ and μ_2 is an extension of μ_1. In accordance with Zorn's lemma a maximal element (E,μ) in $\mathcal{F}$ exists.

If $E \ne C(X,K)$, there exists $g \in C(X,K) \setminus E$. Let $E_- := \{f : f \in E, f \le g\}$ and $E_+ := \{f : f \in E, g \le f\}$, then $\mu(h) \le \mu(q)$ for each $h \in E_-$ and $q \in E_+$, consequently, an element $b \in K$ exists such that $\mu(E_-) \le b \le \mu(E_+)$. Then we put $F = E \cup \{g+g^c, g^c+g : c \in K\}$ (or F is a minimal left or right module over K containing E and g correspondingly). Then one can put $\mu(g+g^c) = b+c$ and $\mu(g^c+g) = c+b$ (moreover, $\mu(d(g+g^c)) = d\mu(g)+dc$ or $\mu((g+g^c)d) = \mu(g)d+cd$ for each $d \in K$ correspondingly) for each $c \in K$. Then μ is an order preserving weakly additive functional (left or right homogeneous correspondingly) on F. This contradicts the maximality of A.

For $C_+(X,K)$ the proof is analogous.

15. Definitions. If sets X and Y are given and $f : X \to Y$ is a mapping, K_1, K_2 are ordered quasirings (or may be particularly semirings) with an order-preserving algebraic homomorphism $u : K_1 \to K_2$ then it induces the mapping $\mathcal{O}(f,u) : \mathcal{O}(X,K_1) \to \mathcal{O}(Y,K_2)$ according to the formula:

(1) $(\mathcal{O}(f,u)(\nu))(g) = u[\nu(g_1(f))]$ for each $g_1 \in C(Y,K_1)$ and $\nu \in \mathcal{O}(X,K_1)$, where $u \circ g_1 = g \in C(Y,K_2)$, $g_1 \in C(Y,K_1)$, $(\mathcal{O}(f,u)(\nu))$ is defined on $(\hat{f},\hat{u})(C(X,K_1)) = \{t : t \in C(Y,K_2); \forall x \in X \ t(x) = u(h \circ f(x)), h \in C(Y,K_1)\}$.

By $I(f,u)$ will be denoted the restriction of $\mathcal{O}(f,u)$ onto $I(X,K)$. If K is fixed, i.e. $u = id$, then we write for short $\mathcal{O}(f)$ and $I(f)$ omitting $u = id$. If

$X=Y$ and $f=id$ we write for short $\mathcal{O}_2(u)$ and $I_2(u)$ respectively omitting $f=id$.

Let $\mathcal{S}$ denote a category such that a family $Ob(\mathcal{S})$ of its objects consists of all sets, a family of morphisms $Mor(X,Y)$ consists of all mappings $f: X\to Y$ for every $X,Y\in Ob(\mathcal{S})$.

Let $\mathcal{K}$ be the category objects of which $Ob(\mathcal{K})$ are all ordered quasirings satisfying Conditions 2.14, $Mor(A,B)$ consists of all order-preserving algebraic homomorphisms for each $A,B\in\mathcal{K}$. Then by $\mathcal{K}_w$ we denote its subcategory of well-ordered quasirings and their order-preserving algebraic homomorphisms.

We denote by $\mathcal{OK}$ a category with the families of objects $Ob(\mathcal{OK}) = \{\mathcal{O}(X,K): X\in Ob(\mathcal{S}), K\in Ob(\mathcal{K}_w)\}$ and morphisms $Mor(\mathcal{O}(X,K_1),\mathcal{O}(Y,K_2))$ for every $X,Y\in Ob(\mathcal{S})$ and $K_1,K_2\in Ob(\mathcal{K}_w)$. Then $\mathcal{IK}$ denotes a category with families of objects $Ob(\mathcal{IK})=\{I(X,K): X\in Ob(\mathcal{S}), K\in Ob(\mathcal{K}_w)\}$ and morphisms $Mor(I(X,K_1),I(Y,K_2))$ for every $X,Y\in Ob(\mathcal{S})$ and $K_1,K_2\in Ob(\mathcal{K})$.

By $\mathcal{S}_l$ will be denoted a category objects of which are linearly ordered sets, $Mor(X,Y)$ consists of all monotone nondecreasing mappings $f: X\to Y$, that is $f(x)\le f(y)$ for each $x\le y\in X$, where $X,Y\in Ob(\mathcal{S}_l)$. Then analogously $\mathcal{O}_l(f,u): \mathcal{O}_l(X,K_1)\to\mathcal{O}_l(Y,K_2)$ for each $X,Y\in Ob(\mathcal{S}_l)$ and $f\in Mor(X,Y)$, $K_1,K_2\in Ob(\mathcal{K})$, $u\in Mor(K_1,K_2)$ according to the formula:

(2) $(\mathcal{O}_l(f,u)(\nu))(g)=u[\nu(g_1(f))]$ for each $g_1\in C_+(Y,K_1)$ and $u\circ g_1=g\in C(Y,K_2)$ and $\nu\in\mathcal{O}_l(X,K_1)$, where $(\mathcal{O}_l(f,u)(\nu))$ is defined on $(\hat{f},\hat{u})(C_+(X,K_1)) := \{t:\ t\in C_+(Y,K_2); \forall x\in X\ t(x)=u(h\circ f(x)), h\in C_+(Y,K_1)\}$. Then the category $\mathcal{O}_l\mathcal{K}$ with families of objects $Ob(\mathcal{O}_l\mathcal{K})=\{\mathcal{O}_l(X,K): X\in Ob(\mathcal{S}_l), K\in Ob(\mathcal{K})\}$ and morphisms $Mor(\mathcal{O}_l(X,K_1),\mathcal{O}_l(Y,K_2))$ and the category $\mathcal{I}_l\mathcal{K}$ with $Ob(\mathcal{I}_l\mathcal{K})=\{I_l(X,K): X\in Ob(\mathcal{S}_l), K\in Ob(\mathcal{K})\}$ and $Mor(I_l(X,K_1),I_l(Y,K_2))$ are defined.

Subcategories of left homogeneous functionals we denote by $\mathcal{O}_h\mathcal{K}$, $\mathcal{O}_{l,h}\mathcal{K}$, $\mathcal{I}_h\mathcal{K}$, $\mathcal{I}_{l,h}\mathcal{K}$ correspondingly. They are taken on subcategories $\mathcal{K}_{w,l}$ in $\mathcal{K}$ or $\mathcal{K}_l$ in $\mathcal{K}$ of left distributive quasirings.

16. Lemma. *There exist covariant functors $\mathcal{O}$, $\mathcal{O}_h$ and $\mathcal{O}_l$, $\mathcal{O}_{l,h}$ in the categories $\mathcal{S}$ and $\mathcal{S}_l$ respectively.*

Proof. If $X,Y\in Ob(\mathcal{S})$ and $f\in Mor(X,Y)$, $g\le h$ in $C(Y,K)$, where $K\in Ob(\mathcal{K}_w)$ (or in $\mathcal{K}_{w,l}$) is marked, then $g\circ f\le h\circ f$ in $C(X,Y)$, consequently, $(\mathcal{O}(f)(\nu))(g)=\nu(g\circ f)\le\nu(h\circ f)=(\mathcal{O}(f)(\nu))(h)$ for each $\nu\in\mathcal{O}(X,K)$. If $c\in K$, $g^c\in C(Y,K)$, then $g^c\circ f\in C(X,K)$, also $(\mathcal{O}(f)(\nu))(g^c+h)=\nu(g^c\circ f+h\circ f)=c+\nu(h\circ f)=c+(\mathcal{O}(f)(\nu))(h)$ and $(\mathcal{O}(f)(\nu))(h+g^c)=\nu(h\circ f+g^c\circ f)=\nu(h\circ f)+c=(\mathcal{O}(f)(\nu))(h)+c$ for each $h\in C(Y,K)$. If $1_X\in Mor(X,X)$, $1_X(x)=x$ for each $x\in X$, then $1_X\circ q=q$ for each $q\in Mor(Y,X)$ and $t\circ 1_X=t$ for each $t\in Mor(X,Y)$. On the other hand, $(\mathcal{O}(1_X)(\nu))(g)=\nu(g\circ 1_X)=\nu(g)$ for each $g\in C(X,K)$, i.e. $\mathcal{O}(1_X)=1_{\mathcal{O}(X)}$. Evidently, $(\mathcal{O}(f\circ s)(\nu))(g)=\nu(g\circ f\circ s)=(\mathcal{O}(s)(\nu)(g\circ f)=((\mathcal{O}(f)\circ\mathcal{O}(s))(\nu))(g)$.

If $\nu \in \mathcal{O}_h(X,K)$, then $(\mathcal{O}(f)(\nu))(bg) = \nu(bg \circ f) = b\nu(g \circ f) = (b(\mathcal{O}(f)(\nu))(g)$. For $\mathcal{O}_l$ (or $\mathcal{O}_{l,h}$) the proof is analogous with $X, Y \in Ob(\mathcal{S}_l)$, $C_+(X,K)$ and $C_+(Y,K)$, where $K \in Ob(\mathcal{K})$ (or $K \in Ob(\mathcal{K}_l)$) is marked.

17. Proposition. *If $f \in Mor(X,Y)$ for $X, Y \in Ob(\mathcal{S})$ or in $Ob(\mathcal{S}_l)$, then*

$\mathcal{O}(f)(I(X,K)) \subseteq I(Y,K)$ and $\mathcal{O}_h(f)(I_h(X,K)) \subseteq I_h(Y,K)$ for $K \in Ob(\mathcal{K}_{w,l})$ or $\mathcal{O}_l(f)(I_h(X,K)) \subseteq I_l(Y,K)$ or $\mathcal{O}_{l,h}(f)(I_{l,h}(X,K)) \subseteq I_{l,h}(Y,K)$ for $K \in Ob(\mathcal{K})$ or $K \in Ob(\mathcal{K}_l)$ correspondingly.

Proof. If $g, h \in C(Y,K)$ are such that $g \vee h$ or $g \wedge h$ exists (see Condition (3) in Lemma 6) and $f : X \to Y$ is a mapping, $\nu \in I(X,K)$ (or $I_l(X,K)$), then

$(\mathcal{O}(f)(\nu))(g \vee h) = \nu(g \circ f \vee h \circ f) = \nu(g \circ f) \vee \nu(h \circ f) = (\mathcal{O}(f)(\nu))(g) \vee (\mathcal{O}(f)(\nu))(h)$ or

$(\mathcal{O}(f)(\nu))(g \wedge h) = \nu(g \circ f \wedge h \circ f) = \nu(g \circ f) \wedge \nu(h \circ f) = (\mathcal{O}(f)(\nu))(g) \wedge (\mathcal{O}(f)(\nu))(h)$. Then for each $c \in K$ one gets

$(\mathcal{O}(f)(\nu))(g^c \bigodot h) = \nu(g^c \circ f \bigodot h \circ f) = \nu(g^c \circ f) \bigodot \nu(h \circ f) = c \bigodot (\mathcal{O}(f)(\nu))(h)$ and

$(\mathcal{O}(f)(\nu))(h \bigodot g^c) = \nu(h \circ f \bigodot g^c \circ f) = \nu(h \circ f) \bigodot \nu(g^c \circ f) = (\mathcal{O}(f)(\nu))(h) \bigodot c$.

If $\nu \in I_h(X,K)$ (or $I_{l,h}(X,K)$), then $(\mathcal{O}(f)(\nu))(bg) = \nu(bg \circ f) = b\nu(g \circ f) = (b(\mathcal{O}(f)(\nu)))(g)$.

18. Definitions. A covariant functor $F : \mathcal{S} \to \mathcal{S}$ will be called epimorphic (monomorphic) if it preserves epimorphisms (monomorphisms). If $\phi : A \hookrightarrow X$ is an embedding, then $F(A)$ will be identified with $F(\phi)(F(A))$.

If for each $f \in Mor(X,Y)$ and each subset A in Y, the equality $(F(f)^{-1})(F(A)) = F(f^{-1}(A))$ is satisfied, then a covariant functor F is called preimage-preserving. When $F(\bigcap_{j \in J} X_j) = \bigcap_{j \in J} F(X_j)$ for each family $\{X_j : j \in J\}$ of subsets in $X \in Ob(\mathcal{S})$ (or in $Ob(\mathcal{S}_l)$), the monomorphic functor F is called intersection-preserving.

If a functor F preserves inverse mapping system limits, it is called continuous.

A functor is said to be semi-normal when it is monomorphic, epimorphic, also preserves intersections, preimages and the empty space.

If a functor is monomorphic, epimorphic, also preserves intersections and the empty space, then it is called weakly semi-normal.

19. Proposition. *The functor $\mathcal{O}$ (or $\mathcal{O}_h$, $\mathcal{O}_l$, $\mathcal{O}_{l,h}$) is monomorphic.*

Proof. Consider $X, Y \in Ob(\mathcal{S})$ (or in $Ob(\mathcal{S}_l)$ respectively) with an embedding $s : X \hookrightarrow Y$ (order-preserving respectively). Suppose that $\nu_1 \neq \nu_2 \in \mathcal{O}(X,K)$ (or in $\mathcal{O}_h(X,K)$, $\mathcal{O}_l(X,K)$, $\mathcal{O}_{l,h}(X,K)$ correspondingly). This means that a mapping $g \in C(X,K)$ (or in $C_+(X,K)$ correspondingly) exists such that $\nu_1(g) \neq \nu_2(g)$. A function $u \in C(Y,K)$ (or in $C_+(Y,K)$ respectively) exists such that $u \circ s = g$, hence $(\mathcal{O}(s)(\nu_k)))(u) = \nu_k(u \circ s) = \nu_k(g)$. Thus $\mathcal{O}(s)(\nu_1) \neq \mathcal{O}(s)(\nu_2)$ (or $\mathcal{O}_h(\nu_1) \neq \mathcal{O}_h(\nu_2)$, $\mathcal{O}_l(\nu_1) \neq \mathcal{O}_l(\nu_2)$, $\mathcal{O}_{l,h}(\nu_1) \neq \mathcal{O}_{l,h}(\nu_2)$ correspondingly).

20. Corollary. *The functors I, I_h, I_l and $I_{l,h}$ are monomorphic.*

Proof. This follows from Proposition 19.

21. Proposition. *The functors $\mathcal{O}$, $\mathcal{O}_h$, $\mathcal{O}_l$ and $\mathcal{O}_{l,h}$ are epimorphic.*

Proof. Suppose that $f : X \to Y$ is a surjective mapping, $\nu \in \mathcal{O}(Y, K)$ (or in $\mathcal{O}_h(Y, K)$, $\mathcal{O}_l(Y, K)$, $\mathcal{O}_{l,h}(Y, K)$ respectively). The set L of all mappings $g \circ f : X \to K$ with $g \in C(Y, K)$ (or in $C_+(Y, K)$ correspondingly) is the $\mathcal{A}$-subset or the left module over K in $C(X, K)$ (or in $C_+(X, K)$). Put $\mu(g \circ f) = \nu(g)$. This functional has an extension from L to a functional $\mu \in \mathcal{O}(X, K)$ (or in $\mathcal{O}_h(X, K)$, $\mathcal{O}_l(X, K)$, $\mathcal{O}_{l,h}(X, K)$ correspondingly) due to Lemma 14.

22. Lemma. *Let L be a submodule over K of $C(X, K)$ or $C_+(X, K)$ relative to the operations $\vee$, $\wedge$, $\bigodot$ and containing all constant mappings $g^c : X \to K$, where $c \in K$. Let also $\nu : L \to K$ be an idempotent (left homogeneous) functional. For each $f \in C(X, K) \setminus L$ or $C_+(X, K) \setminus L$ there exists an idempotent (left homogeneous) extension μ_M of ν on a minimal submodule M containing L and f.*

Proof. For each $g \in M$ one puts

(1) $\nu(g) = \inf\{\nu(h) : h \leq g, h \in L\}$.

Therefore, $\nu(g_1) \leq \nu(g_2)$ for each $g_1 \leq g_2 \in M$. Then

$\nu(g^c \bigodot g) = \inf\{\nu(h) : h \in L, g^c \bigodot g \geq h\} =$

$\inf\{\nu(g^c \bigodot q) : q \in L, g^c \bigodot g \geq g^c \bigodot q\} = c \bigodot \inf\{\nu(q) : q \in L, q \leq g\} = c \bigodot \nu(g)$ and

$\nu(g \bigodot g^c) = \inf\{\nu(h) : h \in L, g \bigodot g^c \geq h\} = \inf\{\nu(q \bigodot g^c) : q \in L, q \bigodot g^c \leq g \bigodot g^c\}$

$= \inf\{\nu(q) : q \in L, q \leq g\} \bigodot c = \nu(g) \bigodot c$.

On the other hand for each $g_1, g_2 \in M$ one gets

$\nu(g_1) \vee \nu(g_2) = \inf\{\nu(g) : g \in L, g_1 \geq g\} \vee \inf\{\nu(q) : q \in L, g_2 \geq q\}$

$= \inf\{\nu(g) \vee \nu(q) : g, q \in L, g_1 \geq g, g_2 \geq q\} \geq \inf\{\nu(g \vee q) : g, q \in L, g_1 \vee g_2 \geq g \vee q\} = \nu(g_1 \vee g_2)$.

From the inequalities $g_k \leq g_1 \vee g_2$ for $k = 1$ and $k = 2$ it follows, that $\nu(g_k) \leq \nu(g_1 \vee g_2)$, consequently, $\nu(g_1) \vee \nu(g_2) = \nu(g_1 \vee g_2)$. Then

$\nu(g_1) \wedge \nu(g_2) = \inf\{\nu(g) : g \in L, g_1 \geq g\} \wedge \inf\{\nu(q) : q \in L, g_2 \geq q\}$

$= \inf\{\nu(g) \wedge \nu(q) : g, q \in L, g_1 \geq g, g_2 \geq q\} \leq \inf\{\nu(g \wedge q) : g, q \in L, g_1 \wedge g_2 \geq g \wedge q\} = \nu(g_1 \wedge g_2)$.

But $\nu(g_k) \geq \nu(g_1 \wedge g_2)$, since $g_k \geq g_1 \wedge g_2$ for $k = 1$ and $k = 2$, consequently, $\nu(g_1) \wedge \nu(g_2) = \nu(g_1 \wedge g_2)$. If ν is left homogeneous, then $\inf\{\nu(bh) : bh \leq bg, h \in L\} = \inf\{\nu(bh) : h \leq g, h \in L\} = b \inf\{\nu(h) : h \leq g, h \in L\}$ for each $b \in K$, consequently, ν is left homogeoeneous on M.

23. Lemma. *If suppositions of Lemma 22 are satisfied, then there exists an idempotent (left homogeneous) functional λ on $C(X, K)$ or $C_+(X, K)$ respectively such that $\lambda|_L = \nu$.*

Proof. The family of all extensions (M, μ_M) of ν on submodules of $C(X, K)$ or $C_+(X, K)$ respectively is partially ordered by inclusion: $(M, \mu_M) \leq (N, \mu_N)$ if and only if $M \subset N$ and $\nu_N|_M = \nu_M$. In view of the Kuratowski-Zorn lemma [6] there exists the maximal submodule P in $C(X, K)$ or $C_+(X, K)$ correspondingly and an idempotent extension ν_P of ν on P. If $P \neq C(X, K)$ or $C_+(X, K)$ correspondingly by Lemma 22 this functional ν_P could be

extended on a module L containing P and some $g \in C(X,K) \setminus P$ or in $C(X,K)_+ \setminus P$ respectively. This contradicts the maximality of (P,ν_P). Thus $P = C(X,K)$ or $C_+(X,K)$ correspondingly.

24. Proposition. *The functors I, I_l and I_h, $I_{l,h}$ are epimorphic.*

Proof. Let $f : X \to Y$ be epimorphic. We consider the set L of all mappings $g \circ f : X \to K$ such that $g \in C(Y,K)$ or $C_+(Y,K)$. Then L is a submodule of $C(X,K)$ or $C_+(X,K)$ relative to the operations $\vee$, $\wedge$, $\bigodot$ and L contains all constant mappings $g^c : X \to K$, where $c \in K$. Put $\mu(g \circ f) = \nu(g)$ for $\nu \in I(X,K)$ or in $I_l(X,K)$, $I_h(X,K)$ or $I_{l,h}(X,K)$. In view of Lemma 23 there is an extension of μ from L onto $C(Y,K)$ or $C_+(Y,K)$ such that $\mu \in I(Y,K)$ or in $I_l(Y,K)$, $I_h(Y,K)$ or $I_{l,h}(Y,K)$ correspondingly.

25. Definition. It is said that $\nu \in \mathcal{O}(X,K)$ (or $\nu \in \mathcal{O}_l(X,K)$) is supported on a subset E in X, if $\nu(f) = 0$ for each $f \in C(X,K)$ or in $C_+(X,K)$ such that $f|_E \equiv 0$. A support of ν is the intersection of all subsets in X on which ν is supported.

26. Proposition. *Let $\nu \in \mathcal{O}(X,K)$ or in $\mathcal{O}_l(X,K)$. Then ν is supported on $E \subset X$ if and only if $\nu(f) = \nu(g)$ for each $f,g \in C(X,K)$ or in $C_+(X,K)$ correspondingly such that $f|_E \equiv g|_E$. Moreover, E is a support of ν if and only if ν is supported on E and for each proper subset F in E, i.e. $F \subset E$ with $F \ne E$, there are $f,h \in C(X,K)$ or in $C_+(X,K)$ respectively with $f|_F \equiv h|_F$ such that $\nu(f) \ne \nu(h)$.*

Proof. Consider $\nu \in \mathcal{O}(X,K)$ such that $\nu(f) = \nu(g)$ for each functions $f,g : X \to K$ with $f|_E = g|_E$. A functional ν induces a functional $\lambda \in \mathcal{O}(E,K)$ such that $\lambda(h) = \nu(h)$ for each $h \in C(X,K)$ with $h|_{X\setminus E} = 0$. Denote by id the identity embedding of E into X. Each function $t : E \to K$ has an extension on X with values in K. Then $\mathcal{O}(id)(\lambda) = \nu$, since $\nu(g^0) = 0$ and hence $\nu(s) = 0$ for each $s \in C(X,K)$ such that $s|_E \equiv 0$.

If $\nu \in \mathcal{O}(X,K)$ and ν is supported on E, then by Definition 25 there exists a functional $\lambda \in \mathcal{O}(E,K)$ such that $\mathcal{O}(id)(\lambda) = \nu$. Therefore $\nu(f) = \lambda(f|_E) = \lambda(g|_E) = \nu(g)$ for each functions $f,g \in C(X,K)$ such that $f|_E = g|_E$.

If E is a support of ν, then by the definition this implies that ν is supported on E. Suppose that $F \subset E$, $F \ne E$ and for each $f,g \in C(X,K)$ with $f|_F \equiv g|_F$ the equality $\nu(f) = \nu(g)$ is satisfied, then a support of ν is contained in F, hence E is not a support of ν. This is the contradiction, hence there are $f,g \in C(X,K)$ with $f|_F \equiv g|_F$ such that $\nu(f) \ne \nu(g)$.

If ν is supported on E and for each proper subset F in E there are $f,h \in C(X,K)$ with $f|_F \equiv h|_F$ such that $\nu(f) \ne \nu(h)$, then ν is not supported on any such proper subset F, consequently, each subset G in X on which ν is supported contains E, i.e. $E \subset G$. Thus E is the support of ν.

One can put

(1) $t_1(x) = \sup\{t(y) : y \in E, y \le x\}$ for each $x \in X$. In the case $\mathcal{O}_l$ the proof is analogous, since each nondecreasing mapping $t : E \to K$ has a nondecreasing extension t_1 on X with values in K, when $X \in Ob(\mathcal{S}_l)$, $E \subset X$.

27. Proposition. *The functors $\mathcal{O}$, I, $\mathcal{O}_l$, I_l, $\mathcal{O}_{l,h}$, $I_{l,h}$ preserve intersections.*

Proof. If E is a subset in X, then there is the natural embedding $C(E,K) \hookrightarrow C(X,K)$ (or $C_+(E,K) \hookrightarrow C_+(X,K)$, when $X \in Ob(\mathcal{S}_l)$ due to Formula 26(1)). Therefore, $\mathcal{O}(E \cap F, K) \subset \mathcal{O}(E,K) \cap \mathcal{O}(F,K)$ (or $\mathcal{O}_l(E \cap F, K) \subset \mathcal{O}_l(E,K) \cap \mathcal{O}_l(F,K)$ respectively). For any subsets E and F in X and each functions $f, g \in C(X,K)$ (or $C_+(X,K)$) with $f|_{E\cap F} \equiv g|_{E\cap F}$ there exists a function $h \in C(X,K)$ (or $C_+(X,K)$) such that $h|_E = f$ and $h|_F = g$. Therefore $\nu(f) = \nu(h)$ and $\nu(g) = \nu(h)$ for each $\nu \in \mathcal{O}(E,K) \cap \mathcal{O}(F,K)$ (or in $\mathcal{O}_l(E,K) \cap \mathcal{O}_l(F,K)$). In view of Proposition 26 the functors $\mathcal{O}$ and $\mathcal{O}_l$ preserve intersections. This implies that the functors I, I_l, $\mathcal{O}_{l,h}$ and $I_{l,h}$ also have this property.

28. Proposition. *Let $\{X_b; p^b_a; V\} =: P$ be an inverse system of sets X_b, where V is a directed set, $p^b_a : X_b \to X_a$ is a mapping for each $a \le b \in V$, $p_b : X = \lim P \to X_b$ is a projection. Then the mappings*

(1) $s = (\mathcal{O}(p_b) : b \in V) : \mathcal{O}(X,K) \to \mathcal{O}(P,K)$ *and* $s_h = (\mathcal{O}_h(p_b) : b \in V) : \mathcal{O}_h(X,K) \to \mathcal{O}_h(P,K)$

(2) $t = (I(p_b) : b \in V) : I(X,K) \to I(P,K)$ *and* $t_h = (I_h(p_b) : b \in V) : I_h(X,K) \to I_h(P,K)$

are bijective and surjective algebraic homomorphisms. Moreover, if $X_b \in Ob(\mathcal{S}_l)$ and p^b_a is order-preserving for each $a < b \in V$, then the mappings

(3) $s_l = (\mathcal{O}_l(p_b) : b \in V) : \mathcal{O}_l(X,K) \to \mathcal{O}_l(P,K)$ *and* $s_{l,h} = (\mathcal{O}_{l,h}(p_b) : b \in V) : \mathcal{O}_{l,h}(X,K) \to \mathcal{O}_{l,h}(P,K)$

(4) $t_l = (I_l(p_b) : b \in V) : I_l(X,K) \to I_l(P,K)$ *and* $t_{l,h} = (I_{l,h}(p_b) : b \in V) : I_{l,h}(X,K) \to I_{l,h}(P,K)$

also are bijective and surjective algebraic homomorphisms.

Proof. We consider the inverse system $\mathcal{O}(P) = (\mathcal{O}(X_a); \mathcal{O}(p^a_b); V\}$ and its limit space $Y = \lim \mathcal{O}(P)$. Then $\mathcal{O}(p^b_a)\mathcal{O}(p_b) = \mathcal{O}(p_a)$ for each $a \le b \in V$, since $p^b_a \circ p_b = p_a$. Let $q : \mathcal{O}(X,K) \to Y$ denote the limit map of the inverse mapping system $q = \lim\{\mathcal{O}(p_a); \mathcal{O}(p^a_b); V\}$.

A functional ν is in $\mathcal{O}(X,K)$ if and only if $\mathcal{O}(p_a)(\nu) \in \mathcal{O}(X_a,K)$ for each $a \in V$, since

(5) $f \in C(X,K)$ if and only if $f = \lim\{f_b; p^b_a; V\}$ and

(6) $\mathcal{O}(p_a)(\nu)(f_a) = \nu(f_a \circ p_a) = \nu_a(f_a)$, where $\nu_a \in \mathcal{O}(X_a,K)$, $f_b \in C(X_b,K)$, $f_b = f_a \circ p^b_a$ for each $a \le b \in V$, $p^b_b = id$, $f(x) = \{f_a \circ p_a(x) : a \in V\} \in \theta(K)$ for each $x = \{x_a : a \in V\} \in X$, where $\{x_a : a \in V\}$ is a thread of P such that $x_a \in X_a$, $p^b_a(x_b) = x_a$ for each $a \le b \in V$, $\theta : K \to K^X$ is an order-preserving algebraic embedding, $\theta(K)$ is isomorphic with K.

If $\nu, \lambda \in \mathcal{O}(X,K)$ are two different functionals, this means that a function $f \in C(X,K)$ exists such that $\nu_1(f) \ne \nu_2(f)$. This is equivalent to the following: there exists $a \in V$ such that $(\mathcal{O}(p_a)(\nu))(f) \ne (\mathcal{O}(p_a)(\lambda))(f)$. Thus the mappings s and analogously t are surjective and bijective.

On the other hand,

(7) $\nu_b(f_b \vee g_b) = \nu_b(f_b) \vee \nu_b(g_b)$ and

(8) $\nu_b(f_b \wedge g_b) = \nu_b(f_b) \wedge \nu_b(g_b)$ for each $b \in V$ and each $\nu_b \in I(X_b,K)$

and every $f_b, g_b \in C(X_b, K)$ such that either $f_b(x) < g_b(x)$ or $f_b(x) = g_b(x)$ or $g_b(x) < f_b(x)$ for each $x \in X_b$, also

(9) $\nu_b(g^c \bigodot f_b) = c \bigodot \nu_b(f_b)$ and

(10) $\nu_b(f_b \bigodot g^c) = \nu_b(f_b) \bigodot c$ for each $c \in K$ and $f_b \in C(X_b, K)$. Taking the inverse limit in Equalities $(5-10)$ gives the corresponding equalities for $\nu \in I(X, K)$, where $\nu = \lim\{\nu_a; I(p^b_a); V\}$, hence t is the algebraic homomorphism.

Analogously s preserves Properties $(9, 10)$, that is $\lambda = \lim\{\lambda_a; \mathcal{O}(p^b_a); V\}$ is weakly additive, where $\lambda_b \in \mathcal{O}(X_b, K)$ for each $b \in V$. Suppose that $f \le g \in C(X, K)$, then $f_b \le g_b$ for each $b \in V$ due to (5). From $\lambda_b(f_b) \le \lambda_b(g_b)$ for each $b \in V$, the inverse limit decomposition $\lambda = \lim\{\lambda_b; \mathcal{O}(p^b_a); V\}$ and (6) it follows that λ is order-preserving.

If $X_b \in Ob(\mathcal{S}_l)$ for each $b \in V$, then X is linearly ordered: $x = \{x_b : b \in V\} \le y = \{y_b : b \in V\}$ if and only if $x_b \le y_b$ for each $b \in V$, where $x, y \in X$ are threads of the inverse system P such that $p^b_a(x_b) = x_a$ for each $a \le b \in V$. Since p^b_a is order-preserving for each $a \le b \in V$ and each f_b is non-decreasing, then f is nondecreasing and hence $f \in C_+(X, K)$ for each $f = \lim\{f_b; p^b_a; V\}$, where $f_b \in C_+(X_b, K)$ and $f_b = f_a \circ p^b_a$ for each $a \le b \in V$ and $x \in X$, $f(x) = \{f_a \circ p_a(x) : a \in V\}$.

Moreover, $\nu \in \mathcal{O}_h(X, K)$ is left homogeneous if and only if $\theta(p_a)(\nu)$ is left homogeneous for each $b \in V$, since $(\mathcal{O}_h(p_a)(\nu))(f_a) = \nu(f_a \circ p_a) = \nu_a(f_a)$.

29. Lemma. *There exist covariant functors $\mathcal{O}_2$, I_2, and $\mathcal{O}_{l,2}$, $I_{l,2}$ and $\mathcal{O}_{h,2}$, $I_{h,2}$ and $\mathcal{O}_{l,h,2}$, $I_{l,h,2}$ in the categories $\mathcal{K}_w$ and $\mathcal{K}$ and $\mathcal{K}_{w,l}$ and $\mathcal{K}_l$ respectively.*

Proof. If $K_1, K_2, K_3 \in Ob(\mathcal{K}_w)$, $u \in Mor(K_1, K_2)$, $v \in Mor(K_2, K_3)$, $\nu \in I(X, K_1)$, then $(I_2(vu)(\nu))(f) = v \circ u \circ \nu(f_1) = [I_2(v)(I_2(u)(\nu))](f)$ for each $f_1 \in C(X, K_1)$ such that $f(x) = v \circ u \circ f_1(x)$ for each $x \in X$, where $X \in Ob(\mathcal{S})$. That is $I_2(vu) = I_2(v)I_2(u)$. Evidently, $I_2(id) = 1$.

If $f(x) \le g(x)$, then $u(f(x)) \le u(g(x))$, where $x \in X$, $f, g \in C(X, K_1)$. Therefore, if $f \vee g$ or $f \wedge g$ exists in $C(X, K_1)$, then $u(f \vee g) = u(f) \vee u(g)$ or $u(f \wedge g) = u(f) \wedge u(g)$ in $C(X, K_2)$ respectively. If $f, g \in C(X, K_1)$, then $u(f(x) + g(x)) = u(f(x)) + u(g(x))$ for each $x \in X$, particularly for $f = g^c$ or $g = g^c$, where $c \in K_1$. Therefore, $u(g^c \bigodot g) = g^{u(c)} \bigodot u(g)$ and $u(g \bigodot g^c) = u(g) \bigodot g^{u(c)}$. To each $\nu_n \in \mathcal{O}(X, K_n)$ and $u \in Mor(K_n, K_{n+1})$ there corresponds a functional $u \circ \nu_n$ on $(\hat{id}, \hat{u})(C(X, K_n))$, $(\hat{id}, \hat{u})(C(X, K_n)) \hookrightarrow C(X, K_{n+1})$ (see §15). If $u : K_n \to K_{n+1}$ is not an epimorphism, the image $(\hat{id}, \hat{u})(C(X, K_n))$ is a proper submodule over $u(K_n)$ in $C(X, K_{n+1})$.

If $K_n, K_{n+1} \in Ob(\mathcal{K})$ and $X \in Ob(\mathcal{S}_l)$, $u \in Mor(K_n, K_{n+1})$, then $u : C_+(X, K_n) \to C_+(X, K_{n+1})$ is a homomorphism. If $K_n, K_{n+1} \in Ob(\mathcal{K}_l)$ and $X \in Ob(\mathcal{S}_l)$ (or $K_n, K_{n+1} \in Ob(\mathcal{K}_{w,l})$ and $X \in Ob(\mathcal{S})$) and $\nu \in \mathcal{O}_h(X, K_n)$ or in $I_h(X, K_n)$, $u \in Mor(K_n, K_{n+1})$, then $u \circ \nu \in \mathcal{O}_h(X, K_{n+1})$ or in $I_h(X, K_{n+1})$ respectively.

This and the definitions above imply that $\mathcal{O}_2(u) : \mathcal{O}(X, K_1) \to \mathcal{O}(X, K_2)$, $I_2(u) : I(X, K_1) \to I(X, K_2)$ and $\mathcal{O}_{l,2}(u)$, $I_{l,2}(u)$ and $\mathcal{O}_{h,2}(u)$, $I_{h,2}(u)$ and $\mathcal{O}_{l,h,2}(u)$, $I_{l,h,2}(u)$ are the homomorphisms. Thus $\mathcal{O}_2 : \mathcal{K}_w \to \mathcal{OK}$ and $\mathcal{O}_{l,2}$:

$\mathcal{K} \to \mathcal{O}_l\mathcal{K}$, $I_2 : \mathcal{K}_w \to \mathcal{I}\mathcal{K}$ and $I_{l,2} : \mathcal{K} \to \mathcal{I}_l\mathcal{K}$, $\mathcal{O}_{h,2} : \mathcal{K}_{w,l} \to \mathcal{O}_h\mathcal{K}$, $I_{h,2} : \mathcal{K}_{w,l} \to \mathcal{I}_h\mathcal{K}$, $\mathcal{O}_{l,h,2} : \mathcal{K}_l \to \mathcal{O}_{l,h}\mathcal{K}$ and $\mathcal{I}_{l,h,2} : \mathcal{K}_l \to \mathcal{I}_{l,h}\mathcal{K}$ are the covariant functors on the categories $\mathcal{K}_w$, $\mathcal{K}$, $\mathcal{K}_{w,l}$ and $\mathcal{K}_l$ correspondingly with values in the categories of (skew) idempotent functionals, when a set $X \in Ob(\mathcal{S})$ or in $Ob(\mathcal{S}_l)$ correspondingly is marked.

30. Proposition. *The bi-functors I on $\mathcal{S} \times \mathcal{K}_w$, I_l on $\mathcal{S}_l \times \mathcal{K}$, I_h on $\mathcal{S} \times \mathcal{K}_{w,l}$ and $I_{l,h}$ on $\mathcal{S}_l \times \mathcal{K}_l$ preserve pre-images.*

Proof. In view of Proposition 24 and Lemma 29 I, I_l, I_h and $I_{l,h}$ are the covariant bi-functors, i.e. the functors in $\mathcal{S}$ or $\mathcal{S}_l$ and the functors in $\mathcal{K}_w$ or $\mathcal{K}$ or $\mathcal{K}_{w,l}$ or $\mathcal{K}_l$ correspondingly as well. For any functor F the inclusion $F(f^{-1}(B)) \subset (F(f))^{-1}(F(B))$ is satisfied.

Suppose the contrary that I does not preserve pre-images. This means that there exist $X, Y \in Ob(\mathcal{S})$ and $K_1, K_2 \in Ob(\mathcal{K}_w)$ or $X, Y \in Ob(\mathcal{S}_l)$ and $K_1, K_2 \in Ob(\mathcal{K})$, $f \in Mor(X, Y)$, $u \in Mor(K_1, K_2)$, $A \subset X$ and $B \subset Y$, $A = F^{-1}(B)$, $\nu \in I(X, K_1)$ such that $I(f, u)(\nu) \in I(B, K_2)$ but $\nu \notin I(f^{-1}(B), u^{-1}(K_2))$ (or $\nu \in I_l(X, K_1)$, $I_l(f, u)(\nu) \in I_l(B, K_2)$ and $\nu \notin I_l(f^{-1}(B), u^{-1}(K_2))$ respectively). One can choose two functions $g, h \in C(X, K_1)$ such that

(1) $g|_A = h|_A$,

(2) $0 < c_1 = u[\inf_{x\in X} g(x)]$, $0 < c_2 = u[\inf_{x\in X} h(x)]$ and

(3) $u[\nu(g)] \neq u[\nu(h)]$.

There exist functions $s, t \in C(X, K_1)$ such that

(4) $s|_A = g|_A$ and $t|_A = h|_A$, while

(5) $s|_{X\setminus A} = t|_{X\setminus A}$ and

(6) $s(x) \le g(x)$ and $s(x) \le h(x)$ for each $x \in X \setminus A$, where g, h satisfy Conditions $(1-3)$. There are also functions $q, r \in C(X, K_1)$ such that

(7) $q|_{X\setminus A} = g|_{X\setminus A}$ and $r|_{X\setminus A} = h|_{X\setminus A}$ with

(8) $q(x) = r(x)$ and $q(x) \le c$ for each $x \in A$, where

(9) $c \in K_1$, $c < \inf_{x\in X} g(x)$, $c < \inf_{x\in X} h(x)$ such that $u(c) < c_1$ and $u(c) < c_2$.

Evidently, $c_1 \le u[\nu(g)]$ and $c_2 \le u[\nu(h)]$. Then

(10) $\nu(g) = \nu(s \vee q) = \nu(s) \vee \nu(q)$ and

(11) $\nu(h) = \nu(t \vee r) = \nu(t) \vee \nu(r)$ and $u[\nu(q)] \neq u[\nu(r)]$.

On the other hand, there are functions $q_1, r_1 \in C(Y, K_2)$, $q_2, r_2 \in C(Y, K_1)$ such that $q_2 \circ f = q$, $r_2 \circ f = r$, $u \circ q_2 = q_1$, $u \circ r_2 = r_1$ and $q_2|_B = r_2|_B$. Therefore, from Properties $(7-10)$ it follows that

(12) $(I(f, u)(\nu))(q_1) = u[\nu(q)] \le u(c)$ and $(I(f, u)(\nu))(r_1) = u[\nu(r)] \le u(c)$. The condition $s = t$ on A and on $X \setminus A$ imply that

(13) $\nu(s) = \nu(t)$. Therefore,

(14) $u(\nu(g)) = u(\nu(s)) \vee u(\nu(q))$ and $u(\nu(h)) = u(\nu(t)) \vee u(\nu(r))$, which follows from $(10, 11)$. But Formulas $(4-6, 12-14)$ contradict the inequality $u[\nu(g)] \neq u[\nu(h)]$, since u is the order-preserving algebraic homomorphism. Thus the bi-functors I and I_l preserve pre-images. The proof in other cases is analogous.

31. Corollary. *If $\nu \in I(X,K)$ or $\nu \in I_l(X,K)$, $f \in Mor(X,Y)$, $u \in Mor(K_1,K_2)$, where $X,Y \in Ob(\mathcal{S})$ and $K_1,K_2 \in Ob(\mathcal{K}_w)$ or $X,Y \in Ob(\mathcal{S}_l)$ and $K_1,K_2 \in Ob(\mathcal{K})$, then $supp(I(f,u)(\nu)) = f(supp(u[\nu]))$ or $supp(I_l(f,u)(\nu)) = f(supp(u[\nu]))$ correspondingly.*

32. Definitions. Suppose that Q is a category and F,G are two functors in Q. Suppose also that a transformation $p : F \to G$ is defined for each $X \in Q$, that is a mapping $p_X : F(X) \to G(X)$ is given. If $p_Y \circ F(f) = G(f) \circ p_X$ for each mapping $f \in Mor(X,Y)$ and every objects $X,Y \in Ob(Q)$, then the transformation $p = \{p_X : X \in Ob(Q)\}$ is called natural.

If $T : Q \to Q$ is an endofunctor in a category Q and there are natural transformations the identity $\eta : 1_Q \to T$ and the multiplication $\psi : T^2 \to T$ satisfying the relations $\psi \circ T\eta = \psi \circ \eta T = 1_T$ and $\psi \circ \psi T = \psi \circ T\psi$, then one says that the triple $\mathbf{T} := (T,\eta,\psi)$ is a monad.

33. Theorem. *There are monads in the categories $\mathcal{S} \times \mathcal{K}_w$, $\mathcal{S}_l \times \mathcal{K}$, $\mathcal{S} \times \mathcal{K}_{w,l}$ and $\mathcal{S}_l \times \mathcal{K}_l$.*

Proof. Let $\bar{g}(\nu) := \nu(g)$ for $g \in C(X,K)$ and $\nu \in I(X,K)$, where $X \in Ob(\mathcal{S})$ and $K \in Ob(\mathcal{K}_w)$. Therefore, $\bar{g} : I(X,K) \to K$. Then

$$\overline{g^b \bigodot g}(\nu) = \nu(g^b \bigodot g) = b \bigodot \nu(g) = b \bigodot \bar{g}(\nu) \text{ and}$$

$$\overline{g \bigodot g^b}(\nu) = \nu(g \bigodot g^b) = \nu(g) \bigodot b = \bar{g}(\nu) \bigodot b,$$

where $g^b(x) = b$ for each $x \in X$, that is

(1) $\overline{g \odot g^b} = \bar{g} \odot \overline{g^b}$ and $\overline{g^b \odot g} = \overline{g^b} \odot \bar{g}$

for each $g \in C(X,K)$ and $b \in K$.

Then we get $\overline{g \vee h}(\nu) = \nu(g \vee h) = \nu(g) \vee \nu(h) = \bar{g}(\nu) \vee \bar{h}(\nu) = (\bar{g} \vee \bar{h})(\nu)$. Moreover, we deduce that $\overline{g \wedge h}(\nu) = \nu(g \wedge h) = \nu(g) \wedge \nu(h) = \bar{g}(\nu) \wedge \bar{h}(\nu) = (\bar{g} \wedge \bar{h})(\nu)$. Thus

(2) $\overline{g \vee h} = \bar{g} \vee \bar{h}$ and $\overline{g \wedge h} = \bar{g} \wedge \bar{h}$.

If additionally ν is left homogeneous and $K \in Ob(\mathcal{K}_{w,l})$, then $\overline{bg} = \nu(bg) = b\nu(g) = b\bar{g}(\nu)$, hence $\overline{bg} = b\bar{g}$ for every $b \in K$ and $g \in C(X,K)$.

For $\lambda \in I(I(X,K),K)$ we put $\xi_{X,K}(\lambda)(g) = \lambda(\bar{g})$ for each $g \in C(X,K)$. Then $\xi_{X,K}(\lambda)(g^b) = \lambda(\overline{g^b}) = \lambda(q^b) = b$, where $q^b : I(X,K) \to K$ denotes the constant mapping $q^b(y) = b$ for each $y \in I(X,K)$. From Formulas (1) it follows that

$$\xi_{X,K}(\lambda)(g^b \bigodot g) = \lambda(\overline{g^b \bigodot g}) = \lambda(b \bigodot \bar{g}) = b \bigodot \lambda(\bar{g}) = b \bigodot \xi_{X,K}(\lambda)(g) \text{ and}$$

$$\xi_{X,K}(\lambda)(g \bigodot g^b) = \lambda(\overline{g \bigodot g^b}) = \lambda(\bar{g} \bigodot \overline{g^b}) = \lambda(\bar{g}) \bigodot b = \xi_{X,K}(\lambda)(g) \bigodot b.$$

On the other hand, from Formulas (2) we get that

$$\xi_{X,K}(\nu)(g \vee h) = \nu(\overline{g \vee h}) = \nu(\bar{g} \vee \bar{h}) = \nu(\bar{g}) \vee \nu(\bar{h}) = \xi_{X,K}(\nu)(g) \vee \xi_{X,K}(\nu)(h) \text{ and}$$

$$\xi_{X,K}(\nu)(g \wedge h) = \nu(\overline{g \wedge h}) = \nu(\bar{g} \wedge \bar{h}) = \nu(\bar{g}) \wedge \nu(\bar{h}) = \xi_{X,K}(\nu)(g) \wedge \xi_{X,K}(\nu)(h)$$

for each $b \in K$, $g,h \in C(X,K)$. Thus $\xi_{X,K} : I(I(X,K),K) \to I(X,K)$.

If $\lambda \in I_h(I_h(X,K),K)$ for $K \in Ob(\mathcal{K}_{w,l})$, then $\xi_{X,K}(\lambda)(bg) = \lambda(\overline{bg}) = \lambda(b\bar{g}) = b\lambda(\bar{g})$, hence $\xi_{X,K} : I_h(I_h(X,K),K) \to I_h(X,K)$. Analogously is defined the mapping $\xi_{X,K} : \mathcal{O}(\mathcal{O}(X,K),K) \to \mathcal{O}(X,K)$ for each $X \in Ob(\mathcal{S})$ and $K \in \mathcal{K}_w$, also $\xi_{X,K} : \mathcal{O}_l(\mathcal{O}_l(X,K),K) \to \mathcal{O}_l(X,K)$, $\xi_{X,K} : I_l(I_l(X,K),K) \to I_l(X,K)$ for each $X \in Ob(\mathcal{S}_l)$ and $K \in \mathcal{K}$, $\xi_{X,K} : I_h(I_h(X,K),K) \to I_h(X,K)$ for $X \in Ob(\mathcal{S})$ and $K \in \mathcal{K}_{w,l}$, $\xi_{X,K} : I_{l,h}(I_{l,h}(X,K),K) \to I_{l,h}(X,K)$ for $X \in Ob(\mathcal{S}_l)$ and $K \in \mathcal{K}_l$. One also puts $\eta : Id_Q \to \mathcal{O}$ or $\eta : Id_Q \to I$ for $Q = \mathcal{S} \times \mathcal{K}_w$, also $\eta : Id_Q \to \mathcal{O}_l$ or $\eta : Id_Q \to I_l$ for $Q = \mathcal{S}_l \times \mathcal{K}$ correspondingly.

Next we verify that the transformations η and ξ are natural for each $f \in Mor(X \times K_1, Y \times K_2)$, i.e. $f = (s,u)$, $s \in Mor(X,Y)$, $u \in Mor(K_1,K_2)$:

$$\eta_{(Y,K_2)} \circ \mathcal{O}((s,u)) = \mathcal{O}(id_Y, id_{K_2})) \circ \mathcal{O}((s,u))$$

$$= \mathcal{O}((s,u)) = \mathcal{O}((s,u)) \circ \mathcal{O}(id_X, id_{K_1})) = \mathcal{O}((s,u)) \circ \eta_{(X,K_1)},$$

$$\xi_{(Y,K_2)} \circ \mathcal{O}((s,u))[\mathcal{O}^2(X,K_1)] = \xi_{(Y,K_2)}(\mathcal{O}(\bar{s},\bar{u})[\mathcal{O}(X,K_1)])$$
$$= \mathcal{O}((s,u)) \circ \eta_{(X,K_1)}[\mathcal{O}^2(X,K_1)]),$$

where $\mathcal{O}^{m+1}(X,K) := \mathcal{O}(\mathcal{O}^m(X,K),K)$ for each natural number m (see also §15 and Proposition 30).

For each $\nu \in \mathcal{O}(X,K)$ and $g \in C(X,K)$ one gets

$$\xi_{X,K} \circ \eta_{(\mathcal{O}(X,K),K)}(\nu)(g) = \eta_{(\mathcal{O}(X,K),K)}(\nu)(\bar{g}) = \bar{g}(\nu) = \nu(g) \text{ and}$$

$$\xi_{X,K} \circ \mathcal{O}(\eta_{(X,K)})(\nu)(g) = (\mathcal{O}(\eta_{(X,K)}(\nu))(\bar{g}) = \nu(\bar{g} \circ \eta_{(X,K)}) = \nu(g).$$

Let now $\tau \in \mathcal{O}^3(X,K)$ and $g \in C(X,K)$, then

$$\xi_{(X,K)} \circ \xi_{\mathcal{O}(X,K)}(\tau)(g) = (\xi_{\mathcal{O}(X,K)}(\tau))(\bar{g}) = \tau(\bar{\bar{g}}) \text{ and}$$

$$\xi_{(X,K)} \circ \mathcal{O}(\xi_{(X,K)})(\tau)(g) = (\mathcal{O}(\xi_{(X,K)})(\tau))(\bar{g}) = \tau(\bar{g} \circ \xi_{(X,K)}) = \tau(\bar{\bar{g}}),$$

where $\bar{\bar{g}} \in C(\mathcal{O}^2(X,K),K)$ is prescribed by the formula $(\bar{\bar{g}})(\nu) = \nu(\bar{g})$ for each $\nu \in \mathcal{O}^2(X,K)$. Thus $\mathbf{O} := (\mathcal{O}, \eta, \xi)$ is the monad. Since I is the restriction of the functor $\mathcal{O}$, the triple $\mathbf{I} := (I, \eta, \xi)$ is the monad in the category $\mathcal{S} \times \mathcal{K}_w$ as well. Analogously $\mathbf{O}_l := (\mathcal{O}_l, \eta, \xi)$ and $\mathbf{I}_l := (I_l, \eta, \xi)$ form the monads in the category $\mathcal{S}_l \times \mathcal{K}$; $\mathbf{O}_h = (\mathcal{O}_h, \eta, \xi)$ and $\mathbf{I}_h = (I_h, \eta, \xi)$ are the monads in $\mathcal{S} \times \mathcal{K}_{w,l}$; $\mathbf{O}_{l,h} = (\mathcal{O}_{l,h}, \eta, \xi)$ and $\mathbf{I}_{l,h} = (I_{l,h}, \eta, \xi)$ are the monads in $\mathcal{S}_l \times \mathcal{K}_l$.

34. Proposition. *If a sequence*

(1) $... \to K_n \to K_{n+1} \to K_{n+2} \to ...$ *in $\mathcal{K}_w$ (or in $\mathcal{K}$) is exact, then sequences*

(2) $... \to \mathcal{O}_2(X,K_n) \to \mathcal{O}_2(X,K_{n+1}) \to \mathcal{O}_2(X,K_{n+2}) \to ...$ *and*

(3) $... \to I_2(X,K_n) \to I_2(X,K_{n+1}) \to I_2(X,K_{n+2}) \to ...$

are exact (analogously for $\mathcal{O}_{l,2}$ and $I_{l,2}$ correspondingly).

Proof. A sequence

$... \to K_n \to K_{n+1} \to K_{n+2} \to ...$ is exact means that $s_n(K_n) = ker(s_{n+1})$ for each n, where $s_n : K_n \to K_{n+1}$ is an order-preserving algebraic homomorphism, $ker(s_{n+1}) = s_{n+1}^{-1}(0)$. Each homomorphism s_n induces the homomorphism $\mathbf{s}_n : C(X, K_n) \to C(X, K_{n+1})$ point-wise $(\mathbf{s}_n(f))(x) = s_n(f(x))$ for each $x \in X$. Therefore, $\mathbf{s}_n(f \vee g) = \mathbf{s}_n(f) \vee \mathbf{s}_n(g)$ or $\mathbf{s}_n(f \wedge g) = \mathbf{s}_n(f) \wedge \mathbf{s}_n(g)$, when $f \vee g$ or $f \wedge g$ exists, where $f, g \in C(X, K_n)$. Moreover, $(\mathbf{s}_n(f+g))(x) = \mathbf{s}_n(f(x)+g(x)) = s_n(f(x)) + s_n(g(x)) = [\mathbf{s}_n(f) + \mathbf{s}_n(g)](x)$ and $[\mathbf{s}_n(fg)](x) = s_n(f(x)g(x)) = s_n(f(x))s_n(g(x)) = [(\mathbf{s}_n(f))(\mathbf{s}_n(g))](x)$, consequently, $\mathbf{s}_n(C(X, K_n)) = \mathbf{s}_{n+1}^{-1}(0)$, since $f_{n+2} \in C(X, K_{n+2})$ is zero if and only if $f_{n+2}(x) = 0$ for each $x \in X$. Thus the sequence

$... \to C(X, K_n) \to C(X, K_{n+1}) \to C(X, K_{n+2}) \to ...$ is exact.

Then a functional $\lambda_{n+2} \in \mathcal{O}(X, K_{n+2})$ is zero on $\mathbf{s}_{n+1}(C(X, K_{n+1}))$ if and only if $\lambda_{n+2}(f_{n+2}) = 0$ for each $f_{n+2} \in \mathbf{s}_{n+1}(C(X, K_{n+1}))$. Therefore, $\mathbf{s}_{n+1}(\lambda_{n+1}) = 0 = \lambda_{n+2}$ on $\mathbf{s}_{n+1}[\mathbf{s}_n(C(X, K_n))]$ if and only if $\lambda_{n+1}(f_{n+1}) \in s_n(K_n)$ for each $f_{n+1} \in \mathbf{s}_n(C(X, K_n))$. But $\mathbf{s}_{n+1}[\mathbf{s}_n(C(X, K_n))] \subset \mathbf{s}_{n+1}(C(X, K_{n+1}))$, consequently, $\mathcal{O}_2(s_n) = ker\mathcal{O}_2(s_{n+1})$. Thus the sequences $(2,3)$ are exact, analogously for other functors I_2, $\mathcal{O}_{l,2}$ and $I_{2,l}$.

35. Lemma. *Let G be a groupoid with the unit acting on a set X such that to each element $g \in G$ a mapping $v_g : X \to X$ corresponds having the properties*

(1) *$v_g v_h = v_{gh}$ for each $g, h \in G$ and*

(2) *$v_e = id$, where $e \in G$ is the unit element, $id(x) = x$ for each $x \in X$. If K is a quasiring with the associative sub-quasiring L, $L \supset \{0, 1\}$, such that*

(3) *$a(bc) = (ab)c$ for each $a, b \in L$ and $c \in K$, a mapping $\rho : G^2 \to L \setminus \{0\}$ satisfies the cocycle condition*

(4) *$\rho(g, x)\rho(h, v_g x) = \rho(gh, x)$ and*

(5) *$\rho(e, x) = 1 \in K$ for each $g, h \in G$ and $x \in X$, then*

(6) *$T_g f(x) := \rho(g, x)\hat{v}_g f(x)$ is a representation of G by mappings T_g of $C(X, K)$ into $C(X, K)$, where $f \in C(X, K)$, $\hat{v}_g f(x) := f(v_g(x))$ for each $g \in G$ and $x \in X$.*

Proof. For each $g, h \in G$ one has $T_g(T_h f(x)) = \rho(g, x)\hat{v}_g[\rho(h, x)\hat{v}_h f(x)] = \rho(gh, x)\hat{v}_{gh} f(x) = T_{gh} f(x)$, hence $T_g T_h = T_{gh}$. Moreover, $T_e f = f$, since $v_e = id$ and $\rho(e, x) = 1$, i.e. $T_e = I$ is the unit operator on $C(X, K)$.

The mappings T_g are (may be) generally non-linear relative to K. If K is commutative, distributive and associative, then T_g are K-linear on $C(X, K)$.

36. Definition. A functional ν on $C(X, K)$ or $C_+(X, K)$ we call semi-idempotent, if it satisfies the property:

(1) $\nu(g + f) = \nu(g) + \nu(f)$ for each $f, g \in C(X, K)$ or $C_+(X, K)$ respectively, where $(g + f)(x) = g(x) + f(x)$ for each $x \in X$.

Suppose that G is a groupoid with the unit acting on a set X and satisfying Conditions $35(1,2)$. A functional λ on $C(X, K)$ or $C_+(X, K)$ we call (T, G)-invariant if

(2) $\hat{T}_g \lambda = \lambda$, where $(\hat{T}_g \lambda)(f) := \lambda(T_g f)$ for each $g \in G$ and f in $C(X, K)$ or $C_+(X, K)$ correspondingly.

Let $S_+(G,K)$ denote the family of all semi-idempotent functionals, when K is commutative and associative relative to the addition for (G,K), let also $S_\vee(G,K)$ (or $S_\wedge(G,K)$) denote the family of all functionals satisfying Conditions 8(4) (or 8(5) correspondingly) for general K. Denote by $H_+(G,K)$ (or $H_\vee(G,K)$ or $H_\wedge(G,K)$) the family of all G-invariant semi-idempotent (or in $S_\vee(G,K)$ or in $S_\wedge(G,K)$ correspondingly) functionals for (X,K), when $X=G$ as a set. We supply these families with the operations of the addition

(3) $\nu(f) +_i \lambda(f) =: (\nu +_i \lambda)(f)$ in $S_j(G,K)$ for $i=1,2,3$ and $j=+,\vee,\wedge$ respectively and the multiplication being the convolution of functionals

(4) $\nu * \lambda(f) = \nu(\lambda(T_g f))$ in $S_j(G,K)$, where $g\in G$, $j\in\{+,\vee,\wedge\}$.

Then we put $H_h(G,K)$, $S_h(G,K)$, $H_{\vee,h}(G,K)$, $S_{\vee,h}(G,K)$, $H_{\wedge,h}(G,K)$ and $S_{\wedge,h}(G,K)$ for the subsets of all left homogeneous functionals in $H_+(G,K)$, $S_+(G,K)$, $H_\vee(G,K)$, $S_\vee(G,K)$, $H_\wedge(G,K)$, $S_\wedge(G,K)$ correspondingly.

37. Proposition. *If ν is a (T,G)-invariant semi-idempotent functional, then its support is contained in $\bigcap_{n=1}^\infty T^n(X)$, where*

$$T(A) := \bigcup_{g\in G} supp(\rho(g,x)\hat{v}_g(\chi_A(x)))$$

for a subset A in X. Moreover, if K has not divisors of zero a support of ν is G-invariant and contained in $\bigcap_{n=1}^\infty P^n(X)$, where

$$P(X) = \bigcup_{g\in G} v_g(X).$$

Proof. If $\nu(f)\ne 0$, then $\nu(T_g f)\ne 0$ for each $g\in G$, when a functional ν is (T,G)-invariant. On the other hand, if $supp(f)\subset supp(\nu)$, then $supp(\rho(g,x)\hat{v}_g f(x))\subset supp(\nu)$. At the same time, $\bigcup_{g\in G} supp(T_g f)\subset \bigcup_{g\in G} supp(\hat{v}_g f)$, since $\rho(g,x)\in L\setminus\{0\}$ for each $g\in G$ and $x\in X$. Taking $f=\chi_{supp(\nu)}$ we get $supp(\nu)\subset T(supp(\nu))\subset T(X)$, hence by induction $supp(\nu)\subset T^n(X)$ for each natural number n.

If K has not divisors of zero, then $supp(\hat{T}_g\nu) = \hat{v}_g supp(\nu)\subset supp(\nu)$ for each $g\in G$, hence $\bigcup_{g\in G}\hat{v}_g supp(\nu) = supp(\nu)$, since $e\in G$ and $\nu_e = id$. That is $supp(\nu)$ is G-invariant. Since $supp(\nu)\subset X$, then $supp(\nu)\subset P(X)$ and by induction $supp(\nu)\subset P^n(X)$ for each natural number n.

38. Proposition. *If G is a groupoid with a unit or a monoid, then $S_+(G,K)$, $S_\vee(G,K)$ and $S_\wedge(G,K)$ for general T_g and K (or $S_h(G,K)$, $S_{h,\vee}(G,K)$ and $S_{h,\wedge}(G,K)$ for $T_g\equiv\hat{v}_g$ or when K is commutative and associative relative to the multiplication) supplied with the convolution 36(4) as the multiplication operation are groupoids with a unit or monoids correspondingly.*

Proof. Certainly, the definitions above imply the inclusion $S_h(G,K)\subset S_+(G,K)$. If $\nu,\lambda\in I_h(G,K)$, then $(\nu*\lambda)(bf) = \nu(\lambda(T_g(bf))) = \nu(b\lambda(T_g f)) = b((\nu*\lambda)(f))$, when $T_g\equiv\hat{v}_g$ or K is commutative and associative relative to the multiplication. We mention that the Dirac functional δ_e belongs to $S_h(G,K)$ and has the property $\nu*\delta_e = \delta_e*\nu = \nu$ for each $\nu\in S(G,K)$, where e is a unit element in G. Thus δ_e is the neutral element in $S(G,K)$.

For a monoid G one has $\hat{v}_s(\hat{v}_u f(x)) = f(s(ux)) = f((su)x) = \hat{v}_{su} f(x)$ for each $f \in C(G,K)$ and $s,u,x \in G$.

If G is a monoid, then $(\nu*(\lambda*\phi))(f) = \nu^u((\lambda*\phi)(T_u f)) = \nu^u(\lambda^s(\phi(T_s T_u f)) = \nu^u(\lambda^s(\phi(T_{su} f))) = (\nu*\lambda)^{su}(\phi(T_{su} f)) = [(\nu*\lambda)*\phi](f)$ for every $f \in C(G,K)$ and $u,s \in G$ and $\nu,\lambda,\phi \in S_j(G,K)$, where $\nu^u(h)$ means that a functional ν on a function h acts by the variable $u \in G$, consequently, $\nu*(\lambda*\phi) = (\nu*\lambda)*\phi$. Thus the family $S_j(G,K)$ is associative, when G is associative, where $j \in \{+,\vee,\wedge,h,(h,\vee),(h,\wedge)\}$ for the corresponding T_g and K.

39. Theorem. *If G is a groupoid with a unit or a monoid, then $S_+(G,K)$ (for K commutative and associative relative to $+$), $S_\vee(G,K)$ and $S_\wedge(G,K)$ for general T_g (or $S_{\vee,h}(G,K)$ and $S_{\wedge,h}(G,K)$ for $T_g \equiv \hat{v}_g$ or when K is commutative and associative relative to the multiplication) are quasirings or semirings correspondingly.*

Proof. If $f,g \in C(X,K)$ or in $C_+(X,K)$ and $f \vee g$ or $f \wedge g$ exists (see Condition (3) in Lemma 6), ν,λ are functionals satisfying Condition 8(4) or 8(5) respectively, then

(1) $(\nu +_i \lambda)(f +_i g) = \nu(f +_i g) +_i \lambda(f +_i g) = (\nu(f) +_i \nu(g)) +_i (\lambda(f) +_i \lambda(g)) = (\nu(f) +_i \lambda(f)) +_i (\nu(g) +_i \lambda(g)) = (\nu +_i \lambda)(f) +_i (\nu +_i \lambda)(g)$
for $i = 1,2,3$, where $+_1 = +$, $+_2 = \vee$, $+_3 = \wedge$. That is, the functional $\nu +_i \lambda$ satisfies Property 36(1) for $i=1$ or 8(4) for $i=2$ or 8(5) when $i=3$ correspondingly. If additionally ν and λ are left homogeneous, then

(2) $(\nu +_i \lambda)(bf) = \nu(bf) +_i \lambda(bf) = b\nu(f) +_i b\lambda(f) = b(\nu +_i \lambda)(f)$
for each $b \in K$.

On the other hand, we deduce that
$((\nu_1 +_i \nu_2)*\lambda)(f) = (\nu_1 +_i \nu_2)(\lambda(T_g f)) = \nu_1(\lambda(T_g f)) +_i \nu_2(\lambda(T_g f))$
$= (\nu_1*\lambda)(f) +_i (\nu_2*\lambda)(f)$ and
$(\lambda*(\nu_1 +_i \nu_2))(f) = \lambda((\nu_1 +_i \nu_2)(T_g f)) = \lambda(\nu_1(T_g f)) +_i \lambda(\nu_2(T_g f))$
$= (\lambda*\nu_1)(f) +_i (\lambda*\nu_2)(f)$
for each $\nu_1,\nu_2,\lambda \in S_j(G,K)$ and $f \in C(G,K)$ or in $C_+(G,K)$ correspondingly, for $i = 1,2,3$ and $i = i(j)$ respectively, where $+_1 = +$, $+_2 = \vee$ and $+_3 = \wedge$. Thus, the right and left distributive rules are satisfied:

(3) $(\nu_1 +_i \nu_2)*\lambda = \nu_1*\lambda +_i \nu_2*\lambda$ and

(4) $\lambda*(\nu_1 +_i \nu_2) = \lambda*\nu_1 +_i \lambda*\nu_2$
for $i = 1,2,3$ respectively.

Therefore, Formulas $(1-4)$ and Proposition 38 imply that $S_+(G,K)$, $S_\vee(G,K)$, $S_\wedge(G,K)$, $S_{\vee,h}(G,K)$ and $S_{\wedge,h}(G,K)$ are left and right distributive quasirings or semirings correspondingly.

40. Theorem. *If G is a groupoid with a unit, $X = G$ as a set (see §36), then $H_j(G,K)$ is an ideal in $S_j(G,K)$, where $j = +$ (for K commutative and associative relative to $+$) or $j = \vee$ or $j = \wedge$ or $j = (\vee,h)$ or $j = (\wedge,h)$ with $\rho(u,x) \equiv 1$; $j = (\vee,h)$ or $j = (\wedge,h)$ for commutative and associative K relative to the multiplication with general T_u.*

Proof. We mention that $\hat{T}_g(b_1\lambda_1 +_i b_2\lambda_2)(f) = b_1\lambda_1(T_g f) +_i b_2\lambda_2(T_g f)$, where the operation denoted by the addition $+_i$ is either $+$ or $\vee$ or $\wedge$ for $i=1$ or $i=2$ or $i=3$ correspondingly (and also below in this section),

consequently, $b_1\lambda_1 +_i b_2\lambda_2 \in H_j(G,K)$ for each $\lambda_1, \lambda_2 \in H_j(G,K)$ and $b_1, b_2 \in K$, $i = i(j)$.

In Formula 36(4) after the action of a functional λ on a function $T_g f(x)$ of the variable x one gets that $\lambda(T_g f) =: h(g)$ is a function in the variable g and ν is acting on this function, i.e. $\nu * \lambda(f) = \nu(h(x))$, where $x, g \in G$. This implies that

$\nu * (\lambda(f +_i t)) = \nu * (\lambda(f) +_i \lambda(t)) = \nu(\lambda(T_g f) +_i \lambda(T_g t))$
$= \nu(\lambda(T_g f)) +_i \nu(\lambda(T_g t)) = (\nu * \lambda)(f) +_i (\nu * \lambda)(t)$ for $i = 1, 2, 3$,

consequently, the convolution operation maps from $S_j(G,K)^2$ into $S_j(G,K)$.

If $\lambda \in H_j(G,K)$ and $\nu \in S_j(G,K)$, then

$(\hat{T}_s(\nu * \lambda))(f) = \hat{T}_s(\nu^u(\lambda^x(T_u f(x)))) = \nu^u(\lambda^x(T_s(T_u f(x)))$
$= \nu^u(\lambda^x(T_u f(x)))) = (\nu * \lambda)(f)$ and
$(\hat{T}_s(\lambda * \nu))(f) = \hat{T}_s(\lambda^u(\nu^x(T_u f(x)))) = \lambda^u(\nu^x(T_s(T_u f(x))))$
$= \lambda^u(T_s(\nu^x(T_u f(x)))) = (\lambda * \nu)(f)$,

since $\lambda^u(T_s g(u)) = \lambda^u(g(u)) = \lambda(g)$, particularly with $g(x) = T_u f(x)$ or $g(u) = \nu^x(T_u f(x))$ correspondingly, whilst $T_s \equiv \hat{v}_s$ in the cases $j = +$ or $j = \vee$ or $j = \wedge$ with $\rho \equiv 1$, or for general $T_u f(x) = \rho(u,x)\hat{v}_s f(x)$ in the cases of homogeneous functionals $j = (\vee, h)$ or $j = (\wedge, h)$ (see §38 also), hence $\nu * \lambda,\ \lambda * \nu \in H_j(G,K)$. Therefore, the latter formula and Theorem 39 imply that

$(\nu +_i H_j(G,K)) * H_j(G,K) \subset (\nu * H_j(G,K)) +_i (H_j(G,K) * H_j(G,K))$
$\subset H_j(G,K) +_i H_j(G,K) \subset H_j(G,K)$ and
$H_j(G,K) * (\nu +_i H_j(G,K)) \subset (H_j(G,K) * \nu) +_i (H_j(G,K) * H_j(G,K))$
$\subset H_j(G,K) +_i H_j(G,K) \subset H_j(G,K)$

for each $\nu \in S_j(G,K)$ and $+_i$ corresponding to j, that is $H_j(G,K)$ is the right and left ideal in $S_j(G,K)$.

References

[1] Yu. A. Bachturin. *Basic structures of modern alegbra* (Moscow: Nauka, 1990).

[2] J.M.G. Fell, R.S. Doran. *Representations of ∗-algebras, locally compact groups, and Banach ∗-algebraic bundles* (Boston: Acad. Press, 1988).

[3] E. Hewitt, K.A. Ross. *Abstract harmonic analysis* (Berlin: Springer-Verlag, 1994).

[4] F. Kasch. *Moduln und Ringe* (Stuttgart: Teubner, 1977).

[5] M. Kilp, U. Knauer, A.V. Mikhalev. *Monoids, acts and categories* (Berlin: De Gruyter, 2000).

[6] K. Kunen. *Set theory* (Amsterdam: North-Holland Publ. Co., 1980).

[7] G.L. Litvinov, V.P. Maslov, G.B. Shpiz. *Idempotent functional analysis: an algebraic approach* // Math. Notes **69: 5-6** (2001), 696-729.

[8] S.V. Ludkovsky. *Topological transformation groups of manifolds over non-Archimedean fields, representations and quasi-invariant measures. I* // J. Mathem. Sci. **147: 3** (2008), 6703-6846.

[9] S.V. Ludkovsky. *Topological transformation groups of manifolds over non-Archimedean fields, representations and quasi-invariant measures, II* // J. Mathem. Sci. **150: 4** (2008), 2123-2223.

[10] E. Mendelson. *Introduction to mathematical logic* (Princeton: D. van Nostrand Co., Inc., 1964).

[11] M.A. Naimark. *Normed rings* (Moscow: Nauka, 1968).

[12] T.Radul. *On the functor of order-preserving functionals* // Comment. Math. Univ. of Carolinae. **39:3** (1998), 609-615.

[13] L. Rowen. *Ring Theory*, Pure and Appl. Math. Ser. **127, 128** (New York: Acad. Press, 1988).

[14] R. D. Schafer. *An introduction to nonassociative algebras* (New York: Acad. Press, 1966).

[15] M.M. Zarichnyi. *Spaces of idempotent measures* // Izvestiya Mathematics **74: 3** (2010), 481-499.